Exotically knotted surfaces

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Content

First exotic phenomena

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The first discovery of exotic phenomena is due to Milnor. He proved:

Theorem (Milnor, 1956)

There exist exotic 7-spheres, i.e. smooth 7-manifolds which are homeomorphic but not diffeomorphic to S^7.

First exotic phenomena

Later on, Milnor and Kervaire were able to compute the exact number of exotic spheres in many dimensions:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\#K_n$	1	1	1	?	1	1	28	2	8	6	992	1	3	2	16256

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What about 4-manifolds?

- Whether there are exotic 4-spheres is an open problem (*smooth Poincaré conjecture in dimension* 4).
- There are uncountably many exotic $\mathbb{R}^4$.
- Many other examples of exotic 4-manifolds are known.

Another example of exotic phenomenon

Definition

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- It is called *topologically slice* if the disk is locally flat.
- It is called *smoothy slice* if the disk is smoothly embedded.

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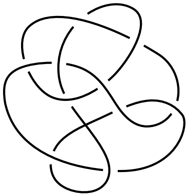
The *Conway knot* is topologically slice, but not smoothly slice.



Goal of the talk

In this talk, we will study the following exotic phenomenon.

Definition

Two smooth proper embeddings $\Sigma^2 \hookrightarrow W^4$ are said to be exotically knotted if they are isotopic through ambient homeomorphisms, but not through ambient diffeomorphisms.

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Definition

Two smooth proper embeddings $\Sigma^2 \hookrightarrow W^4$ are said to be exotically knotted if they are isotopic through ambient homeomorphisms, but not through ambient diffeomorphisms.

Theorem (Hayden, 2021)

There is a pair of exotically knotted disks in B^4 relative boundary.

Goal of the talk

In fact:

Theorem A

Let $n \in \mathbb{N}$. Any compact, connected surface with boundary admits a 2^n-tuple of smooth, proper embeddings in B^4 that are pairwise exotically knotted relative boundary.

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Theorem B

Let $g \geq 0, h \geq 1 \in \mathbb{N}$. The compact, connected, orientable surface with boundary with-h-holes and genus g admits a $(g+1)h$-tuple of smooth, proper embeddings in B^4 that are pairwise exotically knotted.

Content

Handlebody decompositions

Let $0 \leq k \leq n$. The *n-dimensional k-handle* is $h^k = D^k \times D^{n-k}$.

Definition

Let M be an n-manifold with boundary. Attaching a k-handle to M along an embedding $\varphi : S^{k-1} \times D^{n-k} \hookrightarrow \partial M$ consists of gluing h^k to M along φ to obtain the resulting manifold $M \cup_\varphi h^k$.

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Example

The torus $S^1 \times S^1$ admits a handlebody decomposition consisting of one 0-handle, two 1-handles and one 2-handle.

Kirby diagrams

We seek a convenient way to describe handlebodies in dimension 4.

- Start with a single 0-handle D^4.
- Up to isotopy, there is a unique embedding $S^0 \times D^3 \hookrightarrow \partial D^4 = S^3$. So there is a *unique* way to attach 1-handles.

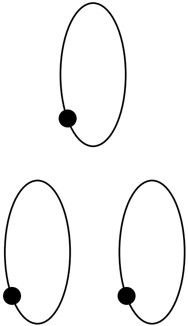


Figure 1: Kirby diagrams for $S^1 \times D^3$ (up) and $(S^1 \times D^3)\#(S^1 \times D^3)$ (down).

Kirby diagrams

- A 2-handle attachment is described by $\varphi : S^1 \times D^2 \hookrightarrow \partial M$. This is determined by
  - a knot $S^1 \times \{0\} \hookrightarrow \partial M$, and
  - a framing of its normal bundle, i.e. an integer.

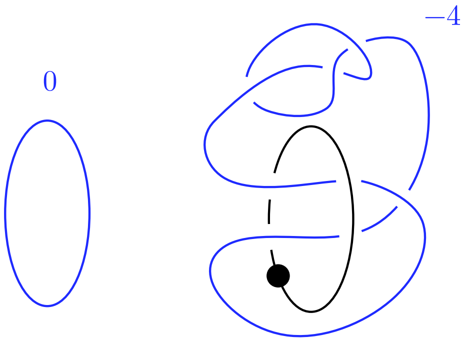
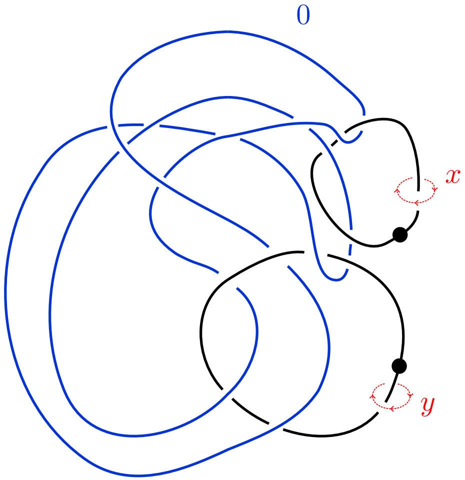


Figure 2: Kirby diagram for $S^2 \times D^2$ (left) and another more complicated Kirby diagram.

Topological invariants from Kirby diagrams

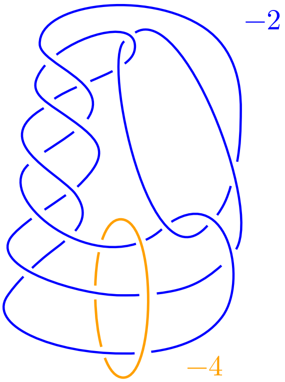
The fundamental group $\pi_1 M$:



$$\pi_1 M \cong \langle x, y | xy^{-1}yx^{-1}y \rangle \cong \langle x, y | y = 1 \rangle \cong \mathbb{Z}.$$

Topological invariants from Kirby diagrams

The intersection form $Q_M : H_2(M) \times H_2(M) \to \mathbb{Z}$:



Q_M is represented by the matrix:

$$\begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}$$

Kirby calculus

Kirby calculus $\equiv$ rules for manipulating Kirby diagrams.

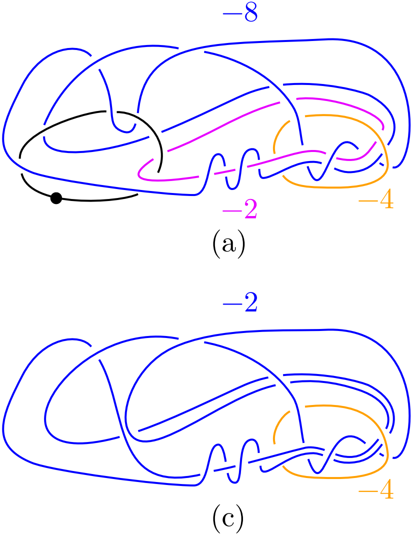


Figure 3: Equivalent Kirby diagrams

Content

The disks $D, D^{'}$

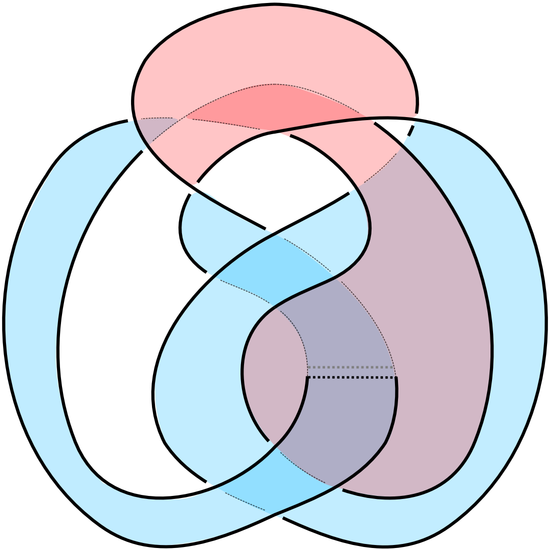


Figure 4: The disk D.

The disks $D, D^{'}$

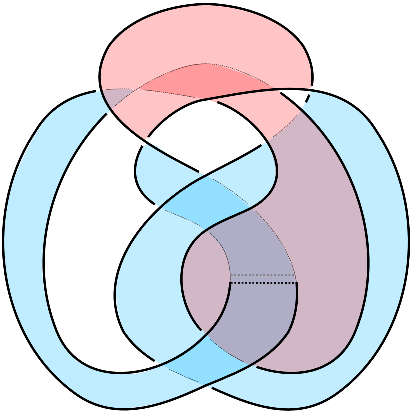


Figure 5: The disks D and D'.

D and D' are topologically isotötopic relative boundary

We use:

Theorem (Conway-Powell, 2021)

If $\pi_1(B^4 \setminus D) \cong \pi_1(B^4 \setminus D') \cong \mathbb{Z}$, then D and D' are topologically isotopic relative boundary.

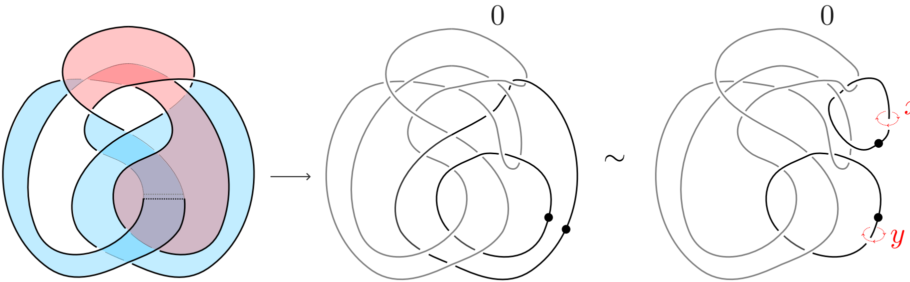


Figure 6: The complement $B^4 \setminus \nu D$.

$$\pi_1(B^4 \setminus \nu D) \cong \langle x, y | xy^{-1}yx^{-1}y \rangle \cong \langle x, y | y = 1 \rangle \cong \mathbb{Z}$$

D and $D^{'}$ are not smoothly isotopic relative boundary

Suppose they are smoothly isotopic relative boundary. Then, so are the annuli

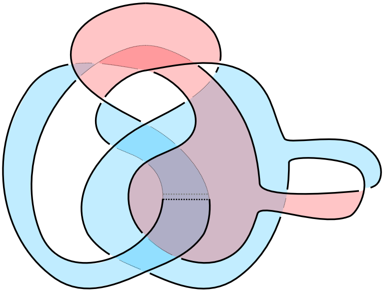


Figure 7: The annuli A (left) and A' (right).

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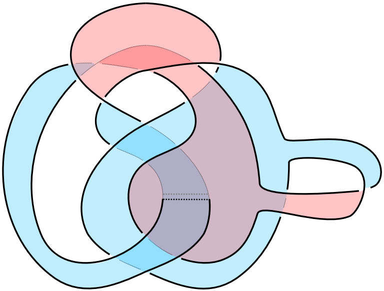

Figure 7: The annuli A (left) and A' (right).

Fact: for F^2 a connected compact surface in B^4, the double branched cover $\Sigma_2(B^4, F^2)$ is unique up to diffeomorphism.

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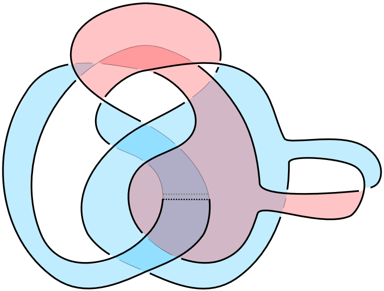


Figure 7: The annuli A (left) and A' (right).

Fact: for F^2 a connected compact surface in B^4, the double branched cover $\Sigma_2(B^4, F^2)$ is unique up to diffeomorphism.
To reach a contradiction, it is enough to prove that $\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A')$ are not diffeomorphic!

$\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A^{'})$ are not diffeomorphic

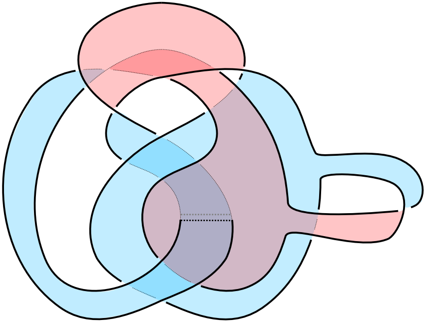


Figure 8: The annuli A (left) and A' (right).

$\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A^{'})$ are not diffeomorphic

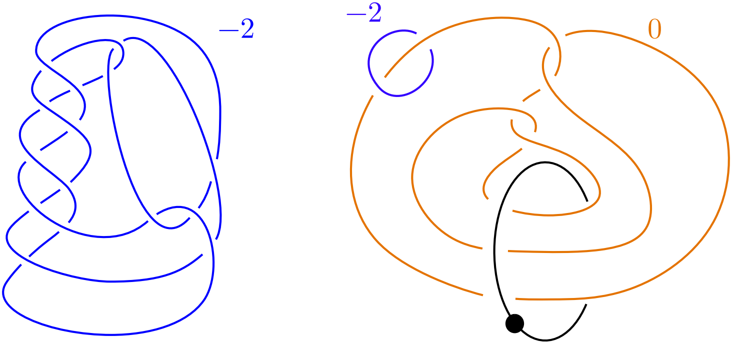


Figure 9: $\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A')$.

$\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A')$ are not diffeomorphic

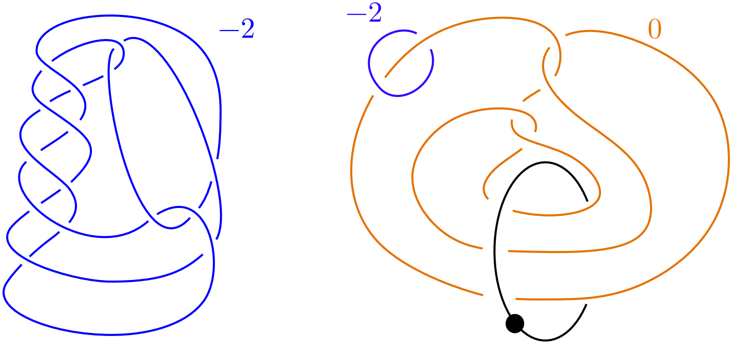


Figure 9: $\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A')$.

Claim: $\Sigma_2(B^4, A')$ contains a smoothly embedded 2-sphere of self-intersection number -2, but $\Sigma_2(B^4, A)$ does not. Hence, they cannot be diffeomorphic.

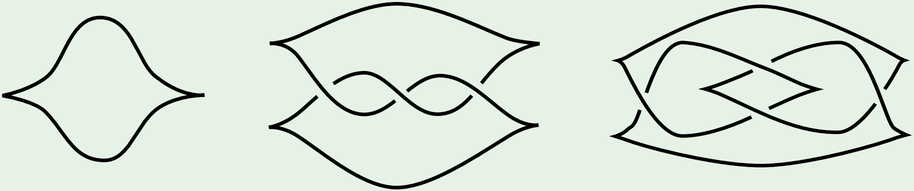
$\Sigma_2(B^4, A)$ does not contain such a 2-sphere

For a Legendrian knot K in (S^3, ξ_{std}), there are two classical invariants:

$$tb(K) = wr(K) - \#\text{right cusps}$$
$$r(K) = \#\text{downward left cusps} - \#\text{upward right cusps}$$

Example 1

The following Legendrian knots have $(tb, r) = (-1, 0), (1, 0), (-3, 0)$.

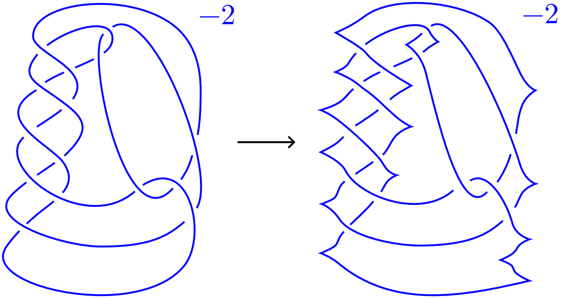


$\Sigma_2(B^4, A)$ does not contain such a 2-sphere

We will need two additional results:

Theorem (Eliashberg, 1990)

Let W^4 be the 4-manifold obtain by attaching a 2-handle h^2 along a Legendrian knot K with framing $tb(K)-1$. Then, W admits the structure of a Stein domain such that $\langle c_1(W), h^2\rangle = r(K)$



We compute: $tb(K) = -1, r(K) = -2$. So W^4 admits a Stein structure with $\langle c_1(W), h^2\rangle = -2$.

$\Sigma_2(B^4, A)$ does not contain such a 2-sphere

Theorem (Lisca-Matić, 1998)

Let W^4 be a Stein domain and S a smoothly embedded 2-sphere with $[S] \neq 0$ in $H_2(W)$. Then, $[S]^2 \leq -2$ and if equality holds, then $\langle c_1(W), [S] \rangle = 0$.

- Suppose there is a smoothly embedded 2-sphere S in $\Sigma_2(B^4, A)$ with $[S]^2 = -2$.
- Then, $[S] = \lambda h^2$ for some $\lambda \neq 0$.
- $0 = \langle c_1(W), [S] \rangle = \lambda \langle c_1(W), h^2 \rangle = -2\lambda \neq 0$, contradiction!

An exotic pair of 4-manifolds

Corollary

$\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A')$ are exotic 4-manifolds.

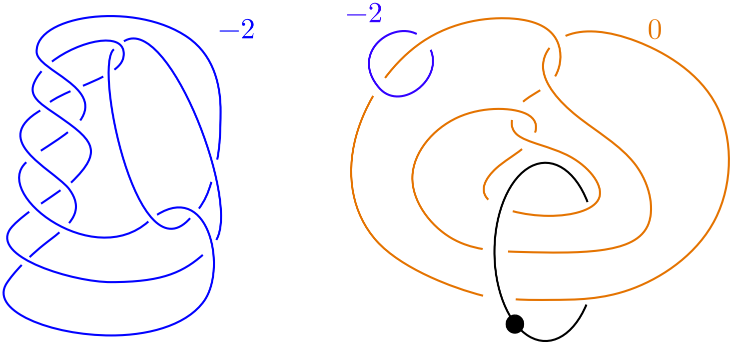


Figure 10: $\Sigma_2(B^4, A)$ and $\Sigma_2(B^4, A')$.

Larger tuples of exotically knotted disks rel boundary

Theorem A

Let $n \in \mathbb{N}$. Any compact, connected surface with boundary admits a 2^n-tuple of smooth, proper embeddings in B^4 that are pairwise exotically knotted relative boundary.

Larger tuples of exotically knotted disks rel boundary

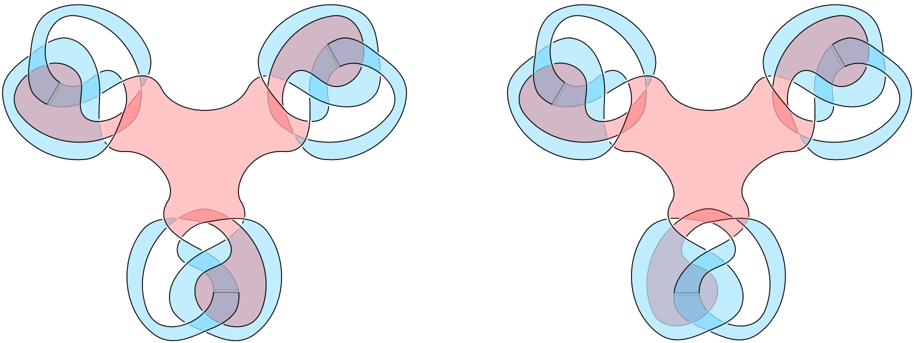


Figure 11: A pair of exotically knotted disks in B^4 relative boundary.

Large tuples of exotically knotted surfaces

Theorem B

Let $g \geq 0, h \geq 1 \in \mathbb{N}$. The compact, connected, orientable surface with boundary with-h-holes and genus g admits a $(g+1)h$-tuple of smooth, proper embeddings in B^4 that are pairwise exotically knotted.

Large tuples of exotically knotted surfaces