Irreducible (conformal) Killing tensors on homogeneous plane waves

Jan Gregorovič

(joint work with L. Zalabova)

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Metric g on (4-dim) manifold M (of Lorentzian signature).
∇ ... Levi-Civita connection

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Projective BGG operators

Depend only on the projective class $[\nabla]$.
$\nabla_{(a}K_{b)} = 0$ ... Killing vectors,
$\nabla_{(a}K_{bc)} = 0$ ... Killing tensors,
$g_{ab}, K_{(a}K'{}_{b)}$ ... reducible Killing tensors for Killing vectors K, K'

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How to find metrics with irreducible (conformal) Killing tensors?

(Conformal) Killing tensors for homogeneous metrics

K ... group of isometries of g acting transitively on $M = K/H$
We described an algebraic method for finding solutions of first BGG operators for projective and conformal geometries:

Gregorovic, J., Zalabova, L., First BGG operators via homogeneous examples, Journal of Geometry and Physics, 192, 2023
Gregorovic, J., Zalabova, L., First BGG operators on homogeneous conformal geometries, Class. Quantum Grav. 40, 2023

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There are distinguished frames corresponding to complement $\mathfrak{c} \subset \mathfrak{k}$ of $\mathfrak{h}$ in which the solutions are described by

- Lie algebra representation $\rho : \mathfrak{k} \to \mathfrak{gl}(V)$,
- linear maps $\pi : V \to \mathfrak{c}$, $\pi : V \to S^2\mathfrak{c}$ or $\pi : V \to S^2_0\mathfrak{c}$

How to compute ρ and π

Step 1 - Find normal projective/conformal Cartan connection

$$\alpha : \mathfrak{k} \to \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

describing the Cartan connection of type (G, P) component-wise

- $\alpha_{-1}(\mathfrak{c}) = \mathfrak{g}_{-1}$,
- $\alpha_0(\mathfrak{c})$ is the connection form of the Levi-Civita connection,
- $\alpha_1(\mathfrak{c})$ is the projective/conformal P-tensor.

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Step 2 - Find prolongation (connection) of the BGG operator

$$\hat{\alpha} = \lambda \circ \alpha + \Phi : \mathfrak{k} \to \mathfrak{gl}(T)$$

on tractor bundle corresponding to representation $\lambda : \mathfrak{g} \to \mathfrak{gl}(T)$.
Φ uniquely obtained by (linearly) normalizing the curvature of $\hat{\alpha}$.

- (c)K vectors, $T = \Lambda^2\mathbb{R}^5$ (dim 10) and $T = \mathfrak{g}$ (dim 15).
- (c)K tensors, $T = S^2\Lambda^2\mathbb{R}^5$ (dim 50) and $T = \boxtimes S^2\mathfrak{g}$ (dim 84).

How to compute ρ and π

Step 3 - Compute ρ and π

$V^0 \subset T$ annihilated by $[\hat{\alpha}(X), \hat{\alpha}(Y)] - \hat{\alpha}([X, Y])$ for all $X, Y \in \mathfrak{k}$.
$V^{i+1} \subset V^i$ ... maximal subset such that $\hat{\alpha}(X)V^{i+1} \subset V^i$ for all $X \in \mathfrak{k}$.
$V = V^{i+1} = V^i$ is Lie algebra representation $\rho = \hat{\alpha}|_V$,
π is the restriction of the natural projection on T to $V \subset T$

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dim(V) ... number of solutions is determined by a rank of overdetermined system of linear equations,
generically all solutions are reducible
$=>$ For algebraically parametrized class of α, the metrics with irreducible solutions form an algebraic subvariety of the parameter space.

The class of homogeneous plane waves

Blau, O'Loughin 2002

Two 3-parameter classes of space-time metrics in Brinkman coordinates (x_+, z_1, z_2, x_-):

$$g_{0,a_1,a_2,\gamma} := 2dx_+dx_- + dz_1^2 + dz_2^2$$
$$+ (z_1, z_2) \exp^{\begin{pmatrix} 0 & -\gamma x_+ \\ \gamma x_+ & 0 \end{pmatrix}} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \exp^{\begin{pmatrix} 0 & \gamma x_+ \\ -\gamma x_+ & 0 \end{pmatrix}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} dx_+{}^2$$
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where $a_1, a_2, \gamma \in \mathbb{R}$.

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where $a_1, a_2, \gamma \in \mathbb{R}$.

$g_{1,a_1,a_2,\gamma}$ is conformally equivalent to $g_{0,a_1+\frac{1}{4},a_2+\frac{1}{4},\gamma}$ and $g_{0,a_1,a_2,\gamma}$ is isometric to $g_{0,\lambda^2 a_1,\lambda^2 a_2,\lambda\gamma}$ and $g_{0,a_2,a_1,\gamma}$

The isometry Lie algebra of plane waves

Killing fields

$\mathfrak{k} = \langle e^1, e^2, e^3, e^4, e^5, e^6 \rangle, \mathfrak{c} = \langle e^1, e^2, e^3, e^4 \rangle, \mathfrak{h} = \langle e^5, e^6 \rangle$ with Lie brackets (for $\epsilon = 0, 1$)

$$[e^1, e^2] = \gamma e^3 + a_1 e^5, [e^1, e^3] = -\gamma e^2 + a_2 e^6, [e^1, e^4] = -\epsilon e^4,$$
$$[e^1, e^5] = e^2 - \epsilon e^5 + \gamma e^6, [e^1, e^6] = e^3 - \gamma e^5 - \epsilon e^6,$$
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Conformal Killing field

There is one additional homothety $e^7 = z_1 \partial_{z_1} + z_2 \partial_{z_2} + 2x_- \partial_{x_-}$.

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There is one additional homothety $e^7 = z_1 \partial_{z_1} + z_2 \partial_{z_2} + 2x_- \partial_{x_-}$.

Conformally flat case

$a_1 = a_2$, $\epsilon = 0, 1 =>$ Conformally flat, γ does not appear in the metric and corresponds to seventh isometry $=> \gamma = 0$

α for homogeneous plane waves

Projective $\alpha : \mathfrak{k} \to \mathfrak{sl}(5, \mathbb{R})$

$$\alpha(x_i e^i) = \begin{bmatrix} 0 & \frac{1}{3}(a_1 + a_2)x_1 & 0 & 0 & 0 \\ x_1 & \epsilon x_1 & 0 & 0 & 0 \\ x_2 & -x_5 & 0 & -\gamma x_1 & 0 \\ x_3 & -x_6 & \gamma x_1 & 0 & 0 \\ x_4 & 0 & x_5 & x_6 & -\epsilon x_1 \end{bmatrix}$$

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Conformal $\beta : \mathfrak{k} \to \mathfrak{so}(2, 4)$

$$\alpha(x_i e^i) = \begin{bmatrix} 0 & \frac{1}{2}(a_1 + a_2)x_1 & 0 & 0 & 0 & 0 \\ x_1 & \epsilon x_1 & 0 & 0 & 0 & 0 \\ x_2 & -x_5 & 0 & -\gamma x_1 & 0 & 0 \\ x_3 & -x_6 & \gamma x_1 & 0 & 0 & 0 \\ x_4 & 0 & x_5 & x_6 & -\epsilon x_1 & -\frac{1}{2}(a_1 + a_2)x_1 \\ 0 & -x_4 & -x_2 & -x_3 & -x_1 & 0 \end{bmatrix}$$

Killing tensors on conformally flat plane waves

Previously investigated by Keane, Tupper 2010 (one mistake).

Symmetric space case $\epsilon = 0, \gamma = 0, a_1 = a_2 = \pm 1$

There are 7 Killing fields, 28 Killing tensors, 1 is irreducible

$$x_+(2\partial_{x_+}\partial_{x_-} + \partial^2_{z_1} + \partial^2_{z_2} - (\pm z_1^2 \pm z_2^2)\partial^2_{x_-}) - (z_1\partial_{z_1} + z_2\partial_{z_2} + 2x_-\partial_{x_-})\partial_{x_-}$$

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Case $\epsilon = 1, \gamma = 0, a_1 = a_2$

There are 7 Killing fields, generically 28 reducible Killing tensors
For $a_1 = a_2 = -\frac{3}{16}$, 34 Killing tensors, 6 of them irreducible

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For $a_1 = a_2 = -\frac{3}{16}$, 34 Killing tensors, 6 of them irreducible
For $a_1 = a_2 = \frac{3}{4}$, 29 Killing tensors, 1 of them irreducible

$$\frac{z_1^2 + z_2^2 - 4x_-x_+}{x_+^2}(2\partial_{x_+}\partial_{x_-} + \partial_{z_1}^2 + \partial_{z_2}^2 - \frac{3}{4}(z_1^2 + z_2^2)\partial_{x_-}^2) - (z_1\partial_{z_1} +$$

$$z_2\partial_{z_2} + 4x_-\partial_{x_-} - 2x_+\partial_{x_+})(\frac{z_1\partial_{z_1} + z_2\partial_{z_2} - 2x_+\partial_{x_+}}{x_+^2} + \frac{z_1^2 + z_2^2}{x_+^3}\partial_{x_-})$$

CK tensors on conformally nonflat plane waves

Symmetric space case $\epsilon = 0, \gamma = 0, a_2 = 2 + a_1$

There are 7 conformal Killing fields, generically 27 reducible conformal Killing tensors
For $a_1 = \frac{2}{3}$ or $a_1 = -\frac{8}{3}$, 36 conformal Killing tensors, 9 of them irreducible

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For $a_1 = \frac{2}{3}$ or $a_1 = -\frac{8}{3}$, 36 conformal Killing tensors, 9 of them irreducible

Case $\epsilon = 0, \gamma = 1$

There are 7 conformal Killing fields, generically 27 reducible conformal Killing tensors
For $a_1 = 3, a_2 = -1$, 29 conformal Killing tensors, 2 of them irreducible
For $256a_1^3 - 2112a_1^2 a_2 + 4608a_1 a_2^2 - 1024a_2^3 - 3271a_1^2 + 2878a_1 a_2 + 1449a_2^2 - 988a_1 + 1192a_2 + 12 = 0$, 28 conformal Killing tensors, 1 of them irreducible

Killing tensors on conformally nonflat plane waves

Symmetric space case $\epsilon = 0, \gamma = 0, a_2 = 2 + a_1$

6 Killing fields, generically 22 Killing tensors, 1 of them irreducible

$$x_+ g^{-1}_{0,0,a_1,2+a_1} - e^7 \partial_{x_-}$$

For $a_1 = 0$, 23 Killing tensors, 2 of them irreducible

$$z_1 g^{-1}_{0,0,0,2} - e^7 \partial_{z_1}; \quad x_+ g^{-1}_{0,0,0,2} - e^7 \partial_{x_-}$$

For $a_1 = \frac{2}{3}$, 23 Killing tensors, 2 of them irreducible

$$x_+ g^{-1}_{0,0,\frac{2}{3},\frac{8}{3}} - e^7 \partial_{x_-}; \qquad z_2 \partial^2_{z_1} - z_1 \partial_{z_1} \partial_{z_2} + \frac{2}{3} z_1^2 z_2 \partial^2_{x_-}$$

For $a_1 = -\frac{8}{3}$, 23 Killing tensors, 2 of them irreducible

$$x_+ g^{-1}_{0,0,-\frac{8}{3},-\frac{2}{3}} - e^7 \partial_{x_-}; \qquad z_1 \partial^2_{z_2} - z_2 \partial_{z_1} \partial_{z_2} - \frac{2}{3} z_1 z_2^2 \partial^2_{x_-}$$

Killing tensors on conformally nonflat plane waves

Case $\epsilon = 0, \gamma = 1$

6 Killing fields, generically 22 Killing tensors, 1 of them irreducible
Potencially, one algebraic subvariety with 23 Killing tensors, 2 of them irreducible

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Case $\epsilon = 1$

6 Killing fields, generically 22 reducible Killing tensors
Without assumption, too hard to analyze.
$\gamma = 0$: for $a_2 = 4a_1 + \frac{3}{4}$, $a_1 = 0$ or $a_2 = \frac{1}{4}a_1 - \frac{3}{16}$, 23 Killing tensors, 1 of them irreducible
$a_1 = 0$: for $\gamma = 0, a_2 = \frac{3}{4}$, 27 Killing tensors, 5 of them irreducible, or for $\gamma = 0$, $a_2 = -\frac{3}{16}$, 28 Killing tensors, 6 of them irreducible