Simplifying harmonic gauge perturbations around black holes

[arXiv:1711.00585, 1801.09800, 2004.09651] + WIP

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Perturbations around Black Holes

- On a Lorentzian (M, g), $R_{\mu\nu} = 0$ vacuum, consider scalar z ($s = 0$), Maxwell v_μ ($s = 1$) and Einstein $p_{\mu\nu}$ ($s = 2$) perturbations:

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\begin{aligned}
&(SW) && \Box z = 0, \\
&(Max)\ (VW) && \Box v_\mu - \nabla_\mu \nabla^\nu v_\nu = 0 \\
& && (v_\mu = \nabla_\mu \varepsilon \rightsquigarrow \Box\varepsilon = 0 \text{ residual gauge dynamics}) \\
&(Ein)\ (LW) && \Box p_{\mu\nu} - 2\,{}^4R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2\,\nabla_{(\mu}\nabla^\lambda \bar{p}_{\nu)\lambda} = 0 \\
& && (p_{\mu\nu} = \nabla_{(\mu} v_{\nu)} \rightsquigarrow \Box v_\mu = 0 \text{ residual gauge dynamics})
\end{aligned}
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- Under harmonic gauges ($\nabla^\mu v_\mu = 0$ and $\nabla^\nu \bar{p}_{\mu\nu} = \nabla^\nu(p_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \operatorname{tr} p) = 0$) we get the vector wave and Lichnerowicz wave equations.
- **Advantages:** well known regularity properties for solutions in harmonic gauge
- **Disadvantages:** reduction to master equations and separation of variables is usually done in Regge-Wheeler (Schwarzschild) or radiation (Kerr) gauges; not obvious in harmonic gauge.

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Schwarzschild background

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Full separation of variables for any $s = 0, 1, 2$:

$$\Phi(t, r, \theta, \varphi) = \{\phi_{\omega lm}(r) Y^{lm}(\theta, \varphi)\} e^{-i\omega t}$$

- Harmonic gauge equations result in complicated, coupled radial mode equations!
- But gauge invariant modes decouple and satisfy spin-s Regge-Wheeler equations $\mathcal{D}_s \phi^s(r) = 0$.

$$\mathcal{D}_s \phi := \partial_r f \partial_r \phi - \frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}\phi + \omega^2 \frac{1}{f}\phi$$

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Radial Mode Equation: $VW_\omega[v] = 0$

Explicitly, $v_\mu \to v(r) = (v_t, v_r, u \mid w)$:

(odd) $$\partial_r \mathcal{B}_l r^2 f \partial_r w + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \mathcal{B}_l w + \mathcal{B}_l \frac{2M}{r} w = 0,$$

(even) $$\begin{bmatrix} -\partial_r \frac{1}{f}\, r^2 f \partial_r\, v_t \\ \partial_r f\, r^2 f \partial_r\, v_r \\ \partial_r \mathcal{B}_l\, r^2 f \partial_r\, u \end{bmatrix} + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \begin{bmatrix} -\frac{1}{f}\, v_t \\ f\, v_r \\ \mathcal{B}_l\, u \end{bmatrix} + i\omega \frac{2M}{f} \begin{bmatrix} v_r \\ -v_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2f^2 & 2\mathcal{B}_l f \\ 0 & 2\mathcal{B}_l f & \mathcal{B}_l \frac{2M}{r} \end{bmatrix} \begin{bmatrix} v_t \\ v_r \\ u \end{bmatrix} = 0,$$

where $f(r) = 1 - \frac{2M}{r}$ and $\mathcal{B}_l = l(l+1)$.

Radial Mode Equation: $LW_\omega[p] = 0$ (odd sector)

Explicitly, $p_{\mu\nu} \to p(r) = (h_{tt}, h_{tr}, h_{rr}, j_t, j_r, K, G \mid h_t, h_r, h_2)$:

$$
\begin{bmatrix} \partial_r(-2\frac{\mathcal{B}_l}{f}\, r^2 f \partial_r)\, h_t \\ \partial_r(2\mathcal{B}_l f\, r^2 f \partial_r)\, h_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\, r^2 f \partial_r)\, h_2 \end{bmatrix} - \mathcal{B}_l \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix}
+ \begin{bmatrix} -4\frac{\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 \\ 0 & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix}
- i\omega \frac{4M}{f} \begin{bmatrix} 0 & -\mathcal{B}_l & 0 \\ \mathcal{B}_l & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix} + \omega^2 \frac{r^2}{f} \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix} = 0
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where $f(r) = 1 - \frac{2M}{r}$, $\mathcal{A}_l = (l-1)l(l+1)(l+2)$ and $\mathcal{B}_l = l(l+1)$

Radial Mode Equation: $LW_\omega[p] = 0$ (even sector)

$$
\begin{bmatrix} \partial_r(-2\,r^2 f\partial_r)\,h_{tr} \\ \partial_r(-2\frac{\mathcal{B}_l}{f}\,r^2 f\partial_r)\,j_t \\ \partial_r(\frac{1}{f^2}\,r^2 f\partial_r)\,h_{tt} \\ \partial_r(f^2\,r^2 f\partial_r)\,h_{rr} \\ \partial_r(2\,r^2 f\partial_r)\,K \\ \partial_r(2\mathcal{B}_l f\,r^2 f\partial_r)\,j_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\,r^2 f\partial_r)\,G \end{bmatrix}
- \mathcal{B}_l
\begin{bmatrix} -2\,h_{tr} \\ -2\frac{\mathcal{B}_l}{f}\,j_t \\ \frac{1}{f^2}\,h_{tt} \\ f^2\,h_{rr} \\ 2\,K \\ 2\mathcal{B}_l f\,j_r \\ \frac{\mathcal{A}_l}{2}\,G \end{bmatrix}
$$

$$
\begin{bmatrix}
\frac{2(f^2+1)}{f} & -4\mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\
-4\mathcal{B}_l & -\frac{4\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4M^2}{2f^3 r^2} & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{r} & \frac{2}{f}\frac{2M}{r} & 0 & 0 \\
0 & 0 & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{e} & \frac{f(\frac{4M^2}{r^2}-8f^2)}{2} & 4f(1-\frac{3M}{r}) & 4\mathcal{B}_l f^2 & 0 \\
0 & 0 & \frac{2}{f}\frac{2M}{r} & 4f(1-\frac{3M}{r}) & -4(1-\frac{4M}{r}) & -4\mathcal{B}_l f & 0 \\
0 & 0 & 0 & 4\mathcal{B}_l f^2 & -4\mathcal{B}_l f & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\
0 & 0 & 0 & 0 & 0 & 2\mathcal{A}_l f & \mathcal{A}_l
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\begin{bmatrix} h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G \end{bmatrix}
$$

$$
-i\omega\frac{4M}{f}
\begin{bmatrix}
0 & 0 & -\frac{1}{f} & -f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\mathcal{B}_l & 0 \\
\frac{1}{f} & 0 & 0 & 0 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 0 & 0 \\
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+ \omega^2\frac{r^2}{f}
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= 0
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Simplified Radial Mode Equations

- Vector wave equation [arXiv:1711.00585]:
  - $VW_\omega^{\text{odd}} \sim \mathcal{D}_1 \quad VW_\omega^{\text{even}} \sim$
- Lichnerowicz wave equation [arXiv:2004.09651]:
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- Hierarchy of modes:
  pure gauge, gauge invariant, constraint violating.
  (see 2004.09651 or youtu.be/dy-QO5NFHC0 for details.)

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Simplification of a Differential Equation

- A simplification is an isomorphism $E[\phi] = 0 \sim \tilde{E}[\tilde{\phi}] = 0$ from a more complicated PDE to a simpler PDE.
- **Q:** What is a(n iso)morphism between Differential Equations?

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$$\tilde{k} \circ k = \mathrm{id} - h \circ E$$

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Simplification of a Differential Equation

- A simplification is an isomorphism $E[\phi] = 0 \sim \tilde{E}[\tilde{\phi}] = 0$ from a more complicated PDE to a simpler PDE.
- **Q:** What is a(n iso)morphism between Differential Equations?

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Kerr background

- **Kerr:** axially symmetric, stationary black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} \;=\; -\frac{\Delta_r}{\Sigma}(\mathrm{d}\tau + y^2 \mathrm{d}\psi)^2 + \frac{\Delta_y}{\Sigma}(\mathrm{d}\tau - r^2 \mathrm{d}\psi)^2 + \Sigma\left(\frac{(\mathrm{d}r)^2}{\Delta_r} + \frac{(\mathrm{d}y)^2}{\Delta_y}\right),$$

  to Boyer-Lindquist coords: $\tau = t - \boldsymbol{a}\varphi$, $y = \boldsymbol{a}\cos\theta$, $\psi = \varphi/\boldsymbol{a}$,

$$\Sigma = r^2 + y^2, \quad \Delta_y = \boldsymbol{a}^2 - y^2, \quad \Delta_r = r(r - 2\boldsymbol{M}) + \boldsymbol{a}^2,$$

  where $\boldsymbol{M}$ — mass, $\boldsymbol{a}$ — angular momentum.
- Partial separation of variables for $s = 1$:

$$\Phi = \phi_{\omega m}(r, y) e^{-i\omega t} e^{im\psi}, \quad \Box\Phi = 0 \quad \rightsquigarrow \quad VW_{\omega m}[\phi(r, y)] = 0$$

- Teukolsky scalars ($\Phi^{\pm 1} = \Phi^{\pm 1}[\phi]$) decouple, $VW[\phi] = 0 \rightsquigarrow \mathcal{T}^{\pm 1}[\Phi^{\pm 1}] = 0$, and the Teukolsky Master Equation fully separates.

$$\Phi^{\pm 1}_{\omega m}(r, y) = R^{\pm 1}_{\omega m\lambda}(r) Y^{\pm 1}_{\omega m\lambda}(y), \quad \mathcal{T}[\Phi] = 0 \;\rightsquigarrow\; \mathcal{T}_r[R(r)] = 0, \mathcal{T}_y[Y(y)] = 0.$$

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Full separation of variables in harmonic gauge?

- On **Kerr** the fully separable Teukolsky Master Equation $\mathcal{T}^{\pm 1}_{\omega m}[\Phi^{\pm 1}] = 0$ accounts only for a special combination of the components of $VW_{\omega m}[\phi] = 0$, $s = 1$ in harmonic gauge.
- **Q:** Could a more sophisticated approach fully separate $VW_{\omega m}[\phi(r, y)] = 0$, like on Schwarzschild?
- Recent work on Hertz potentials on **Kerr** (for $s = 1, 2$) reveals a similar hierarchy of modes as in Schwarzschild: pure gauge, gauge invariant, constraint violating; all "governed" by Teukolsky Master Equations equations.
  [Lunin 1708.06766, Frolov-Krtouš-Kubizňák 1802.09491, Dolan 1906.04808, Dolan-Durkan-Kavanagh-Wardell 2011.03548 2108.06344 2306.16459]

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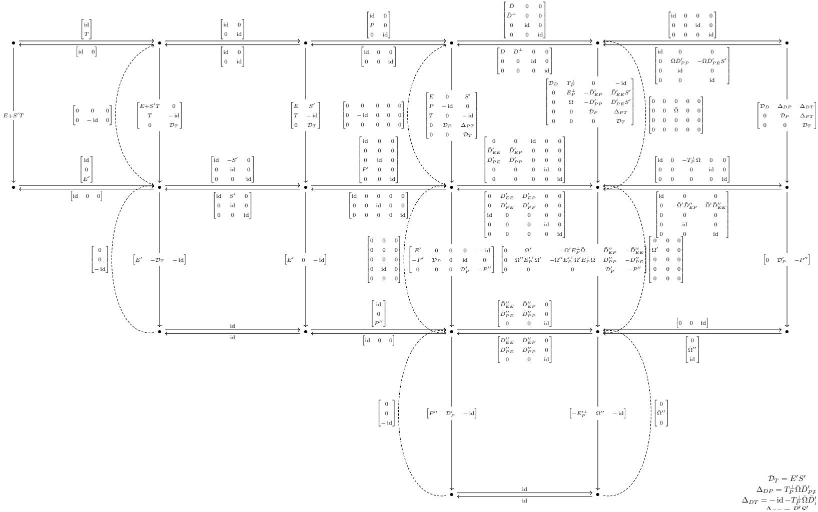

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- pure gauge / gauge invariant modes:
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- **N.B.:** The diagonals $\mathcal{T}^0$ and $\mathcal{T}^{\pm 1}$ all fully separate (in r and y).
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