On Quantitative Aspects of Symplectic Geometry

Daniel Komárek

Charles University

January 24,2025

Motivation

- It is impossible to construct local invariants of symplectic or contact manifolds *(Darboux theorem)*.
- **Symplectic capacities** are global invariants preserved under symplectomorphisms which are monotonic with the respect to symplectic embeddings.

Motivation

- It is impossible to construct local invariants of symplectic or contact manifolds *(Darboux theorem)*.
- **Symplectic capacities** are global invariants preserved under symplectomorphisms which are monotonic with the respect to symplectic embeddings.
- One kind of symplectic capacities arises from the Hutching's **embedded contact homology** – ECH capacities.
- ECH is a variant of Floer homology in dimension three. Applied to show an existence of a closed Reeb orbit in dimension three *(the Weinstein conjecture)* and to provide obstructions of embedding problems.

A **symplectic manifold** (M, ω), where M is a (2n)-dimensional manifold and ω is a symplectic form, i.e. $d\omega = 0$ and ω is non-degenerate.

A **contact manifold** is a $(2n+1)$-dimensional manifold Y equipped with a **contact structure** $\xi := \ker(\lambda)$, where **contact form** λ is a locally defined 1-form such that $\lambda \wedge (d\lambda)^n \neq 0$.

A **symplectic manifold** (M, ω), where M is a (2n)-dimensional manifold and ω is a symplectic form, i.e. $d\omega = 0$ and ω is non-degenerate.

A **contact manifold** is a $(2n+1)$-dimensional manifold Y equipped with a **contact structure** $\xi := \ker(\lambda)$, where **contact form** λ is a locally defined 1-form such that $\lambda \wedge (d\lambda)^n \neq 0$.

A **Reeb vector field** is an unique vector field $R\colon Y \to TY$ such that

$$d\lambda(R, -) = 0, \quad \lambda(R) = 1.$$

A **Reeb orbit** γ is a closed orbit of the flow of R.

From now on (Y, ξ) is a contact 3-manifold.
If γ is passing through a point $p \in Y$, then a linearization of the Reeb flow R along γ determines a symplectic linear map (2x2 matrix)

$$P_\gamma : \xi_p \to \xi_p.$$

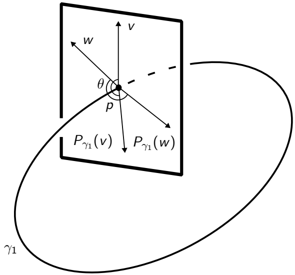
γ is non-degenerate if 1 is not an eigenvalue of P_γ. Assume non-degenerate orbits only.

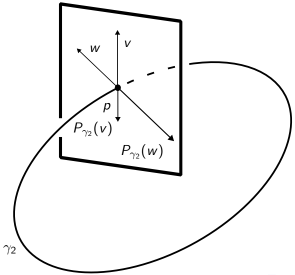
From now on (Y, ξ) is a contact 3-manifold.
If γ is passing through a point $p \in Y$, then a linearization of the Reeb flow R along γ determines a symplectic linear map (2x2 matrix)

$$P_\gamma : \xi_p \to \xi_p.$$

γ is non-degenerate if 1 is not an eigenvalue of P_γ. Assume non-degenerate orbits only.

Eigenvalues ρ, ρ^{-1} of P_γ are either real: γ is **hyperbolic** (positive, or negative), or on the unit circle: γ is **elliptic**.





Pseudoholomorphic curves in symplectizations

An **almost complex structure** on M is an operator $J \in \mathrm{Aut}(TM)$ such that $J^2 = -1$.

An almost complex structure J on a symplectization $\mathbb{R} \times Y$ s.t.

- $J(\partial_s) = R$, where s is the $\mathbb{R}$ coordinate,
- $J(\xi) = \xi$,
- $d\lambda(v, Jv) > 0$ for all $0 \neq v \in \xi$,
- J is invariant under $\mathbb{R}$ action on $\mathbb{R} \times Y$ that translates s,

is called **admissible**.

A **pseudoholomorphic curve** is a differentiable map

$$u\colon (\Sigma, i) \to (\mathbb{R} \times Y, J)$$

s.t. $du \circ i = J \circ du$, where (Σ, i) is a punctured, compact Riemannian surface with a complex structure i and u is assymptotic to closed Reeb orbits at punctures as $\mathbb{R}$ factor goes to $\pm\infty$.

A **pseudoholomorphic curve** is a differentiable map

$$u\colon (\Sigma, i) \to (\mathbb{R} \times Y, J)$$

s.t. $du \circ i = J \circ du$, where (Σ, i) is a punctured, compact Riemannian surface with a complex structure i and u is assymptotic to closed Reeb orbits at punctures as $\mathbb{R}$ factor goes to $\pm\infty$.

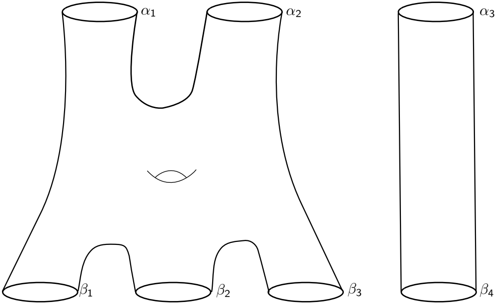


Figure: A pseudoholomorphic curve assymptotic to α_i for $i = 1, 2, 3$ at the positive end and assymptotic to β_j for $j = 1, 2, 3, 4$ at the negative end.

Embedded contact homology

Fix some $\Gamma \in H_1(Y;\mathbb{Z})$. **Orbit set** in class Γ is a set $\alpha := \{(\alpha_i, m_i)\}$ of distinct embedded Reeb orbits, with $m_i = 1$ if α_i is hyperbolic, s.t.

$$\sum_i m_i[\alpha_i] = \Gamma.$$

The chain complex $ECC_*(Y, \lambda, \Gamma, J)$ is generated, as $\mathbb{Z}_2$ modules, by those orbit sets.

Embedded contact homology

Fix some $\Gamma \in H_1(Y; \mathbb{Z})$. **Orbit set** in class Γ is a set $\alpha := \{(\alpha_i, m_i)\}$ of distinct embedded Reeb orbits, with $m_i = 1$ if α_i is hyperbolic, s.t.

$$\sum_i m_i[\alpha_i] = \Gamma.$$

The chain complex $ECC_*(Y, \lambda, \Gamma, J)$ is generated, as $\mathbb{Z}_2$ modules, by those orbit sets.
Let α, β be orbit sets in Γ. A **differential** is defined as follows

$$\partial\alpha := \sum_\beta \#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R})\beta,$$

where $\#$ is signed count and $\mathcal{M}_1(\alpha, \beta)$ is a moduli space of pseudoholomorphic curves in $\mathbb{R} \times Y$ assymptotic at positive ends to α and at negative ends to β such that their **ECH index** is 1, modulo translation.

Denote by $ECH_*(Y, \lambda, \Gamma, J)$ the homology of $ECC_*(Y, \lambda, \Gamma, J)$.
Turns out that ECH does not depend on J or λ, but just ξ.

Denote by $ECH_*(Y, \lambda, \Gamma, J)$ the homology of $ECC_*(Y, \lambda, \Gamma, J)$. Turns out that ECH does not depend on J or λ, but just ξ.

Example. An **ellipsoid** in $\mathbb{C}^2$ for $a, b > 0$ is

$$E(a, b) := \left((z_1, z_2) \in \mathbb{C}^2 \middle| \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right)$$

with $\lambda = \frac{1}{2} \sum_{i=1}^{2} (x_i dy_i - y_i dx_i)$, where $z_i = x_i + y_i$.

The ECH of an ellipsoid is given by

$$ECH_*(\partial E(a, b), \lambda, 0, J) = \begin{cases} \mathbb{Z}_2, & * = 0, 2, 4, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

More about ECH index

If J is regular, α, β two orbit sets and $C \in \mathcal{M}(\alpha, \beta)$ a pseudoholomorphic curve, then

- $I(C) \geq 0$, with an equality iff C is an union of trivial cylinders with multiplicities,
- if $I(C) = 1$, then $C = C_0 \sqcup C_1$, where $I(C_0) = 0$ and C_1 is embedded.

Symplectic capacities

A **symplectic capacity** is an assignment $c\colon \{\text{Symplectic manifolds of dim } 2n\} \to ([0,\infty], \leq)$, satisfying

- $c(M,\omega) \leq c(M',\omega')$ if there is a symplectic embedding $(M,\omega) \to (M',\omega')$,
- $c(M,\alpha\omega) = \alpha c(M,\omega)$ for $\alpha > 0$,
- $0 < c(B^{2n}(1))$ and $c(B^2(1) \times \mathbb{R}^{2n-2}) < \infty$.

Symplectic capacities

A **symplectic capacity** is an assignment
$c\colon \{\text{Symplectic manifolds of dim } 2n\} \to ([0,\infty], \leq)$, satisfying

- $c(M,\omega) \leq c(M',\omega')$ if there is a symplectic embedding $(M,\omega) \to (M',\omega')$,
- $c(M,\alpha\omega) = \alpha c(M,\omega)$ for $\alpha > 0$,
- $0 < c(B^{2n}(1))$ and $c(B^2(1) \times \mathbb{R}^{2n-2}) < \infty$.

Simplest example is a *Gromov radius* of (M,ω):

$$c_B(M,\omega) := \sup\{\alpha > 0 \mid B^{2n}(\alpha) \to (M,\omega)\}.$$

Gromov's non-squeezing theorem imply that

$$c_B(B^2(1) \times \mathbb{R}^{2n-2}) = 1.$$

ECH capacities

Symplectic action of α is $\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda$.
Filtered $ECH_*^L(Y, \lambda, \Gamma)$ is the homology of the ECH chain subcomplex spanned by generators with action less than $L \in \mathbb{R}$.

ECH capacities

Symplectic action of α is $\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda$.
Filtered $ECH_*^L(Y, \lambda, \Gamma)$ is the homology of the ECH chain subcomplex spanned by generators with action less than $L \in \mathbb{R}$.

Full **ECH capacities** of 4-manifold (M, ω) such that $\partial M = Y, d\lambda = \omega|_Y$ are numbers
$c_k(M, \omega) = \inf\{L \in \mathbb{R} \mid dim(Im[ECH^L(Y, \lambda, 0) \to ECH(Y, \lambda, 0)]) \geq k\}$
for $k \in \mathbb{N}$.

ECH capacities do not depend on λ.

ECH capacities

Symplectic action of α is $\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda$.
Filtered $ECH_*^L(Y, \lambda, \Gamma)$ is the homology of the ECH chain subcomplex spanned by generators with action less than $L \in \mathbb{R}$.

Full **ECH capacities** of 4-manifold (M, ω) such that $\partial M = Y, d\lambda = \omega|_Y$ are numbers
$c_k(M, \omega) = \inf\{L \in \mathbb{R} \mid dim(Im[ECH^L(Y, \lambda, 0) \to ECH(Y, \lambda, 0)]) \geq k\}$
for $k \in \mathbb{N}$.

ECH capacities do not depend on λ.

Theorem (M. Hutchings)

If there is symplectic embedding $(M, \omega) \to (int(M'), \omega')$, then

$$c_k(M, \omega) \leq c_k(M', \omega').$$

Full ECH capacities of ellipsoids

Let a/b be an irrational number. Symplectic action on generator is

$$\mathcal{A}(\alpha) = ma + nb,$$

for $m, n \geq 0$.

Dimension of $ECH^L(\partial E(a,b), \lambda, 0)$ in $ECH(\partial E(a,b), \lambda, 0)$ is

$$|\{(m,n) \in \mathbb{N}^2 \mid ma + nb < L\}|.$$

Full ECH capacities of ellipsoids

Let a/b be an irrational number. Symplectic action on generator is

$$\mathcal{A}(\alpha) = ma + nb,$$

for $m, n \geq 0$.
Dimension of $ECH^L(\partial E(a,b), \lambda, 0)$ in $ECH(\partial E(a,b), \lambda, 0)$ is

$$|\{(m,n) \in \mathbb{N}^2 \mid ma + nb < L\}|.$$

Given $T_{a/b}(m,n)$ triangle constisting of major axes and a line with slope $-a/b$ intersecting point (m,n). Then

$$c_k(E(a,b)) = ma + nb, \quad k = |\{T_{a/b}(m,n) \cap \mathbb{N}^2\}|.$$

Full ECH capacities of ellipsoids

Let a/b be an irrational number. Symplectic action on generator is

$$\mathcal{A}(\alpha) = ma + nb,$$

for $m, n \geq 0$.
Dimension of $ECH^L(\partial E(a,b), \lambda, 0)$ in $ECH(\partial E(a,b), \lambda, 0)$ is

$$|\{(m,n) \in \mathbb{N}^2 \mid ma + nb < L\}|.$$

Given $T_{a/b}(m,n)$ triangle constisting of major axes and a line with slope $-a/b$ intersecting point (m,n). Then

$$c_k(E(a,b)) = ma + nb, \quad k = |\{T_{a/b}(m,n) \cap \mathbb{N}^2\}|.$$

Example. For $a > b$:

$$c_1(E(a,b)) = 0, \quad c_2(E(a,b)) = b$$

$$c_3(E(a,b)) = \begin{cases} 2b, & a/b \geq 2, \\ a, & 2 \geq a/b \geq 1, \end{cases}$$

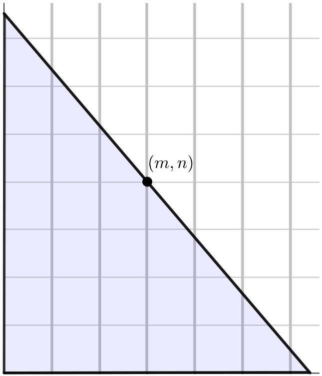


Figure: A triangle $T_{a/b}(m, n)$ bounded by two axes and the line with the slope $-a/b$, intersecting point (m, n).

Let Ω be a convex, compact domain in $[0,\infty]^2$ with a piecewise smooth boundary. A **convex toric domain** is

$$X_\Omega := \{(z_1, z_2) \in \mathbb{C}^2 \mid (\pi|z_1|^2, \pi|z_2|^2) \in \Omega\}.$$

Let Ω be a convex, compact domain in $[0,\infty]^2$ with a piecewise smooth boundary. A **convex toric domain** is

$$X_\Omega := \{(z_1, z_2) \in \mathbb{C}^2 \mid (\pi|z_1|^2, \pi|z_2|^2) \in \Omega\}.$$

It holds

$$c_k(X_\Omega) = \min\{\ell_\Omega(\Lambda) \mid |P_\Lambda \cap \mathbb{Z}^2| = k+1\},$$

where Λ is a convex polygonal path with vertices at lattice points and P_Λ the closed region of Λ.

$\ell_\Omega(\Lambda)$ measures length of polygonal path using dual norm to the one that has translate of Ω as an unit ball.

Example. $c_2(E(1,b) \cap E(c,1)) = R \leq 2$, $c_3(E(1,b) \cap E(c,1)) = 2$ for $b, c > 1$.

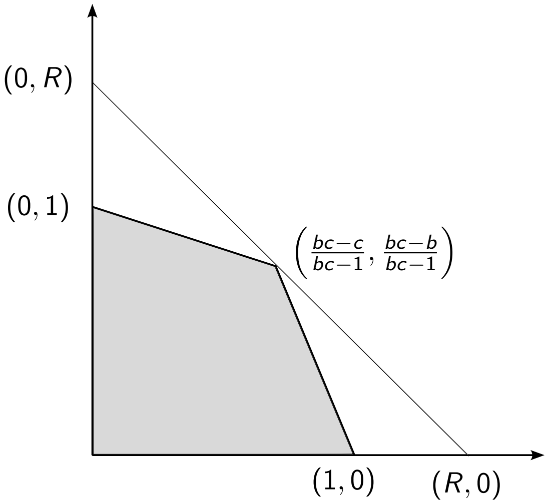


Figure: Ω' of $E(1, b) \cap E(c, 1)$.

Thank you