SuperLie: a package for Lie (super) computations

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45th Winter School Geometry and Physics
Srní
21th Jan 2025

Dedicated to the memory of Pavel Grozman
(18.01.1957–7.03.2022)

SuperLie is a Mathematica(c) package developed by Pavel Grozman.

- Construction of different types pf Lie algebras:
  - $\mathfrak{gl}(V)$
  - Lie (super)algebras with a Cartan matrix
  - Lie (super)algebra of vector fields
    - Poisson, Hamiltonian, contact, etc
  - subalgebra, ideals, quotiens
  - prolongations $(V, \mathfrak{g})_*$
- The highest weight modules and operations on them
- Lie (super)algebras cohomology
- Singular vectors in Verma modules
- Shapovalov form

How to install it?

- The website is

  `https://github.com/andrey-krutov/SuperLie`

- find the releases section

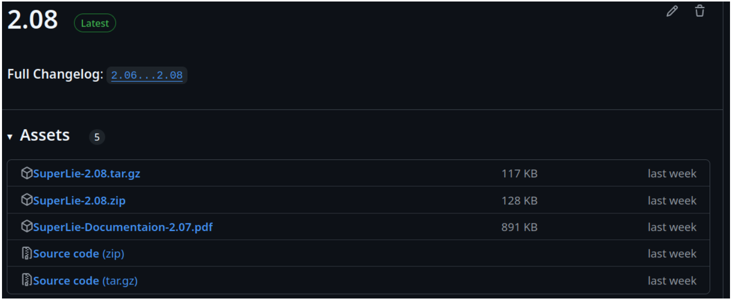
Releases 3

2.08 Latest

last week

+ 2 releases

- download the latest available release and the documentation



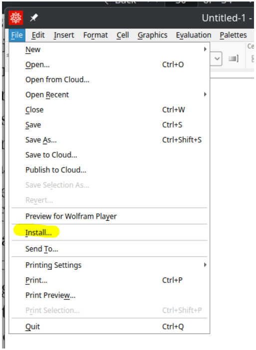
2.08 Latest

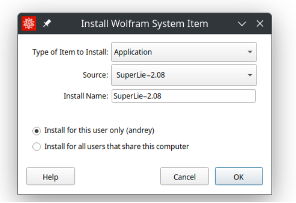
Full Changelog: `2.06...2.08`

Assets 5

SuperLie-2.08.tar.gz	117 KB	last week
SuperLie-2.08.zip	128 KB	last week
SuperLie-Documentaion-2.07.pdf	891 KB	last week
Source code (zip)		last week
Source code (tar.gz)		last week

How to install it?





Type of Item to Install: Application

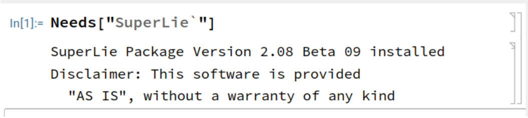
Source: SuperLie-2.08

Install Name: SuperLie-2.08

● Install for this user only (andrey)

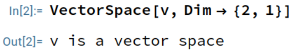
○ Install for all users that share this computer

Load the package



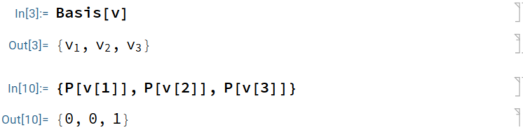
```
In[1]:= Needs["SuperLie`"]

        SuperLie Package Version 2.08 Beta 09 installed
        Disclaimer: This software is provided
          "AS IS", without a warranty of any kind
```

Define a $2|1$-dimensional (super)vector space V



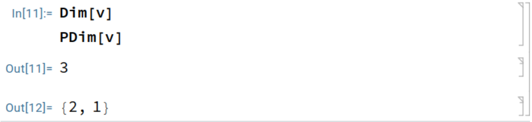
```
In[2]:= VectorSpace[v, Dim → {2, 1}]

Out[2]= v is a vector space
```

Basis vectors and their parities



```
 In[3]:= Basis[v]

Out[3]= {v1, v2, v3}


In[10]:= {P[v[1]], P[v[2]], P[v[3]]}

Out[10]= {0, 0, 1}
```

Dimension and superdimension



```
In[11]:= Dim[v]
         PDim[v]

Out[11]= 3

Out[12]= {2, 1}
```

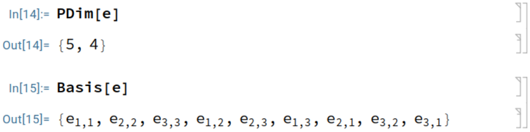
The Lie (super)algebra $\mathfrak{gl}(V)$

The Lie algebra $\mathfrak{gl}(V)$

```
In[13]:= glAlgebra[e, v]
```

Out[13]= e = $\mathrm{gl}(2|1)$

Basis vectors and the dimension



```
In[14]:= PDim[e]
```

Out[14]= $\{5, 4\}$

```
In[15]:= Basis[e]
```

Out[15]= $\{e_{1,1}, e_{2,2}, e_{3,3}, e_{1,2}, e_{2,3}, e_{1,3}, e_{2,1}, e_{3,2}, e_{3,1}\}$

The bracket

```
In[17]:= Act[e[1, 1], e[1, 2]]
```

Out[17]= $e_{1,2}$

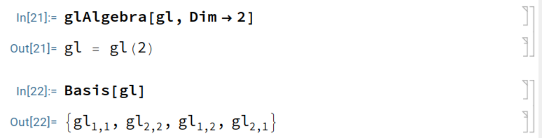
The action of $\mathfrak{gl}(V)$ on V

```
In[18]:= Act[e[1, 2], v[2]]
```

Out[18]= v_1

The Lie (super)algebras $\mathfrak{gl}(n|m)$ and $\mathfrak{sl}(n|m)$

The Lie algebra $\mathfrak{gl}(2)$



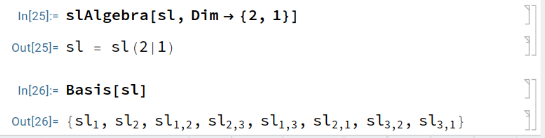
In[21]:= `glAlgebra[gl, Dim → 2]`

Out[21]= `gl = gl(2)`

In[22]:= `Basis[gl]`

Out[22]= $\{\mathtt{gl}_{1,1}, \mathtt{gl}_{2,2}, \mathtt{gl}_{1,2}, \mathtt{gl}_{2,1}\}$

The Lie superalgebra $\mathfrak{sl}(2|1)$



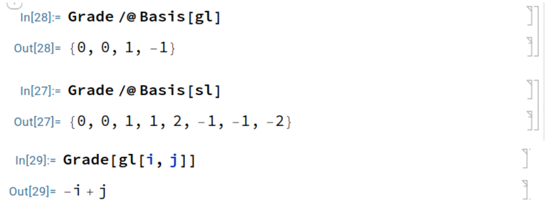
In[25]:= `slAlgebra[sl, Dim → {2, 1}]`

Out[25]= `sl = sl(2|1)`

In[26]:= `Basis[sl]`

Out[26]= $\{\mathtt{sl}_{1}, \mathtt{sl}_{2}, \mathtt{sl}_{1,2}, \mathtt{sl}_{2,3}, \mathtt{sl}_{1,3}, \mathtt{sl}_{2,1}, \mathtt{sl}_{3,2}, \mathtt{sl}_{3,1}\}$

The standard $\mathbb{Z}$-grading



In[28]:= `Grade /@ Basis[gl]`

Out[28]= `{0, 0, 1, -1}`

In[27]:= `Grade /@ Basis[sl]`

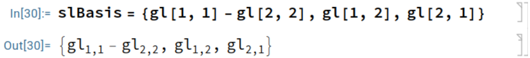
Out[27]= `{0, 0, 1, 1, 2, -1, -1, -2}`

In[29]:= `Grade[gl[i, j]]`

Out[29]= `-i + j`

Subalgebras

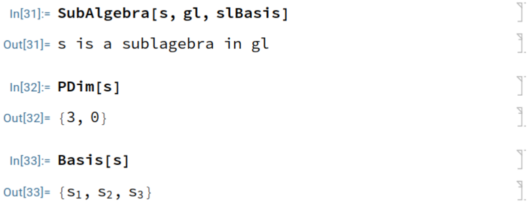
Generators of a subalgebra



```
In[30]:= slBasis = {gl[1, 1] - gl[2, 2], gl[1, 2], gl[2, 1]}
```

Out[30]= $\{gl_{1,1} - gl_{2,2}, gl_{1,2}, gl_{2,1}\}$

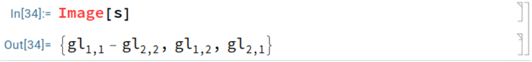
Constructing the subalgebra $\mathfrak{s}$ in $\mathfrak{gl}(2)$



```
In[31]:= SubAlgebra[s, gl, slBasis]

Out[31]= s is a sublagebra in gl

In[32]:= PDim[s]

Out[32]= {3, 0}

In[33]:= Basis[s]
```

Out[33]= $\{s_1, s_2, s_3\}$

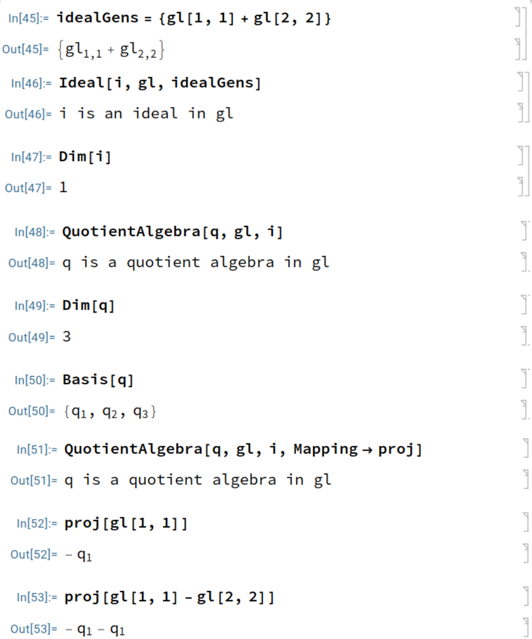
The image of basis elements of $\mathfrak{s}$ inside $\mathfrak{gl}(2)$



```
In[34]:= Image[s]
```

Out[34]= $\{gl_{1,1} - gl_{2,2}, gl_{1,2}, gl_{2,1}\}$

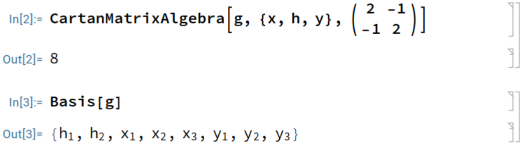
Ideals and quotient algebras



```
In[45]:= idealGens = {gl[1, 1] + gl[2, 2]}

Out[45]= {gl_{1,1} + gl_{2,2}}

In[46]:= Ideal[i, gl, idealGens]

Out[46]= i is an ideal in gl

In[47]:= Dim[i]

Out[47]= 1

In[48]:= QuotientAlgebra[q, gl, i]

Out[48]= q is a quotient algebra in gl

In[49]:= Dim[q]

Out[49]= 3

In[50]:= Basis[q]

Out[50]= {q_1, q_2, q_3}

In[51]:= QuotientAlgebra[q, gl, i, Mapping → proj]

Out[51]= q is a quotient algebra in gl

In[52]:= proj[gl[1, 1]]

Out[52]= - q_1

In[53]:= proj[gl[1, 1] - gl[2, 2]]

Out[53]= - q_1 - q_1
```

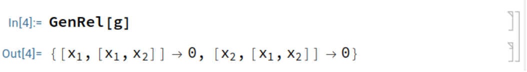
Lie (super)algebras with a Cartan matrix

$\mathfrak{sl}(3)$ from the Cartan matrix: $\mathfrak{g} = \mathfrak{x} \oplus \mathfrak{h} \oplus \mathfrak{y}$



```
In[2]:= CartanMatrixAlgebra[g, {x, h, y}, ( 2  -1
                                           -1   2 )]

Out[2]= 8

In[3]:= Basis[g]

Out[3]= {h1, h2, x1, x2, x3, y1, y2, y3}
```

Relations in $\mathfrak{x}$

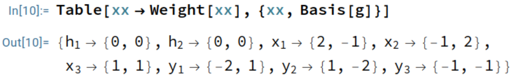


```
In[4]:= GenRel[g]

Out[4]= {[x1, [x1, x2]] → 0, [x2, [x1, x2]] → 0}
```

Basis in $\mathfrak{x}$ in terms of Chevalley generators

```
In[6]:= GenBasis[g]

Out[6]= {x1, x2, [x1, x2]}
```

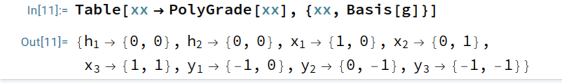
Weights



```
In[10]:= Table[xx → Weight[xx], {xx, Basis[g]}]

Out[10]= {h1 → {0, 0}, h2 → {0, 0}, x1 → {2, -1}, x2 → {-1, 2},
          x3 → {1, 1}, y1 → {-2, 1}, y2 → {1, -2}, y3 → {-1, -1}}
```

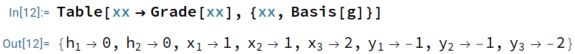
Lie (super)algebras with a Cartan matrix

Roots



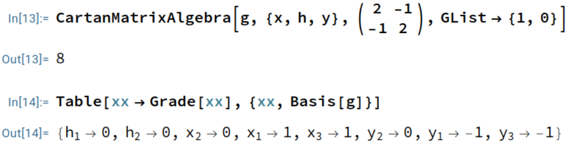
```
In[11]:= Table[xx → PolyGrade[xx], {xx, Basis[g]}]

Out[11]= {h1 → {0, 0}, h2 → {0, 0}, x1 → {1, 0}, x2 → {0, 1},
          x3 → {1, 1}, y1 → {-1, 0}, y2 → {0, -1}, y3 → {-1, -1}}
```

The standard $\mathbb{Z}$-grading



```
In[12]:= Table[xx → Grade[xx], {xx, Basis[g]}]

Out[12]= {h1 → 0, h2 → 0, x1 → 1, x2 → 1, x3 → 2, y1 → -1, y2 → -1, y3 → -2}
```

Non standard $\mathbb{Z}$-gradings



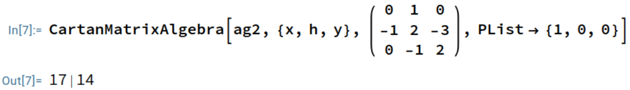
```
In[13]:= CartanMatrixAlgebra[g, {x, h, y}, ( 2  -1), GList → {1, 0}]
                                            (-1   2)

Out[13]= 8

In[14]:= Table[xx → Grade[xx], {xx, Basis[g]}]

Out[14]= {h1 → 0, h2 → 0, x2 → 0, x1 → 1, x3 → 1, y2 → 0, y1 → -1, y3 → -1}
```

Basis of $\mathfrak{g}_i$

```
In[15]:= Basis[g, 0]

Out[15]= {h1, h2, x2, y2}
```

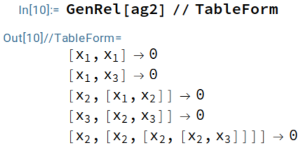
Lie (super)algebras with a Cartan matrix

The exceptional Lie superalgebra $\mathfrak{ag}(2)$ (aka $\mathfrak{g}(3)$)



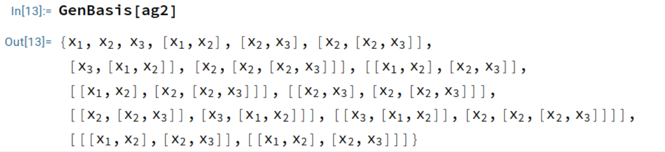
```
In[7]:= CartanMatrixAlgebra[ag2, {x, h, y}, ( 0  1  0
                                             -1  2 -3
                                              0 -1  2 ), PList → {1, 0, 0}]

Out[7]= 17 | 14
```

The relations in $\mathfrak{ag}(2)$



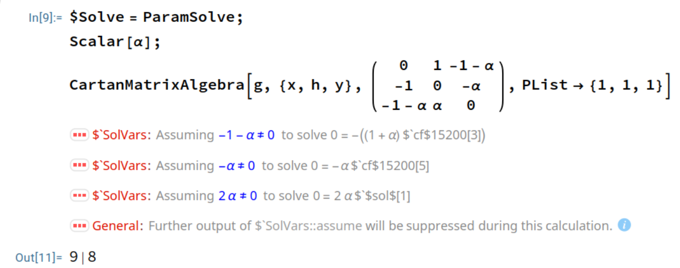
```
In[10]:= GenRel[ag2] // TableForm

Out[10]//TableForm=
        [x1, x1] → 0
        [x1, x3] → 0
        [x2, [x1, x2]] → 0
        [x3, [x2, x3]] → 0
        [x2, [x2, [x2, [x2, x3]]]] → 0
```

The basis in $\mathfrak{ag}(2)$



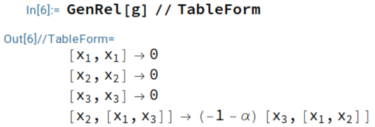
```
In[13]:= GenBasis[ag2]

Out[13]= {x1, x2, x3, [x1, x2], [x2, x3], [x2, [x2, x3]],
          [x3, [x1, x2]], [x2, [x2, [x2, x3]]], [[x1, x2], [x2, x3]],
          [[x1, x2], [x2, [x2, x3]]], [[x2, x3], [x2, [x2, x3]]],
          [[x2, [x2, x3]], [x3, [x1, x2]]], [[x3, [x1, x2]], [x2, [x2, [x2, x3]]]],
          [[[x1, x2], [x2, x3]], [[x1, x2], [x2, x3]]]}
```

Lie superalgebra $\mathfrak{osp}(4|2;\alpha)$



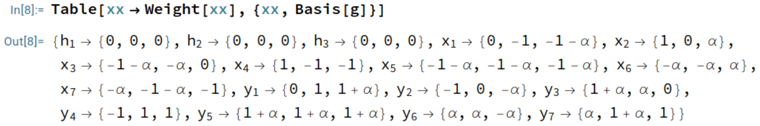
```
In[9]:= $Solve = ParamSolve;
        Scalar[α];
                                                ⎛  0    1  -1 - α ⎞
        CartanMatrixAlgebra[g, {x, h, y},       ⎜ -1    0   -α    ⎟, PList → {1, 1, 1}]
                                                ⎝-1 - α α    0    ⎠

        $`SolVars: Assuming -1 - α ≠ 0  to solve 0 = -((1 + α) $`cf$15200[3])

        $`SolVars: Assuming -α ≠ 0  to solve 0 = -α $`cf$15200[5]

        $`SolVars: Assuming 2 α ≠ 0  to solve 0 = 2 α $`$sol$[1]

        General: Further output of $`SolVars::assume will be suppressed during this calculation.

Out[11]= 9 | 8
```

The relations



```
In[6]:= GenRel[g] // TableForm

Out[6]//TableForm=
        [x1, x1] → 0
        [x2, x2] → 0
        [x3, x3] → 0
        [x2, [x1, x3]] → (-1 - α) [x3, [x1, x2]]
```

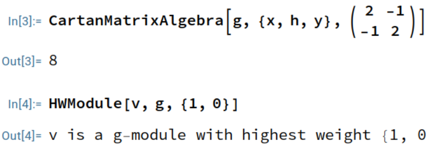
The weights



```
In[8]:= Table[xx → Weight[xx], {xx, Basis[g]}]

Out[8]= {h1 → {0, 0, 0}, h2 → {0, 0, 0}, h3 → {0, 0, 0}, x1 → {0, -1, -1 - α}, x2 → {1, 0, α},
         x3 → {-1 - α, -α, 0}, x4 → {1, -1, -1}, x5 → {-1 - α, -1 - α, -1 - α}, x6 → {-α, -α, α},
         x7 → {-α, -1 - α, -1}, y1 → {0, 1, 1 + α}, y2 → {-1, 0, -α}, y3 → {1 + α, α, 0},
         y4 → {-1, 1, 1}, y5 → {1 + α, 1 + α, 1 + α}, y6 → {α, α, -α}, y7 → {α, 1 + α, 1}}
```

The highest weight modules

The $\mathfrak{sl}(3)$-module V with the highest weight π_1

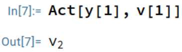


```
In[3]:= CartanMatrixAlgebra[g, {x, h, y}, ( 2  -1 )]
                                          (-1   2 )

Out[3]= 8

In[4]:= HWModule[v, g, {1, 0}]

Out[4]= v is a g-module with highest weight {1, 0}
```

The dimension of V

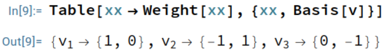
```
In[11]:= PDim[v]

Out[11]= {3, 0}

In[5]:= Basis[v]

Out[5]= {v1, v2, v3}
```

The action



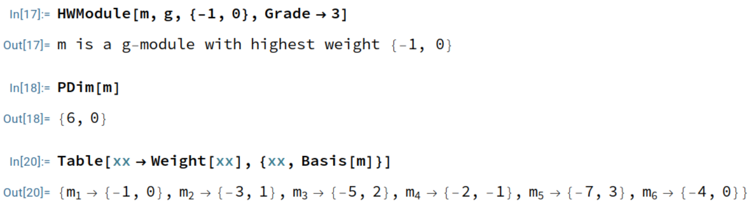
```
In[7]:= Act[y[1], v[1]]

Out[7]= v2
```

The weights



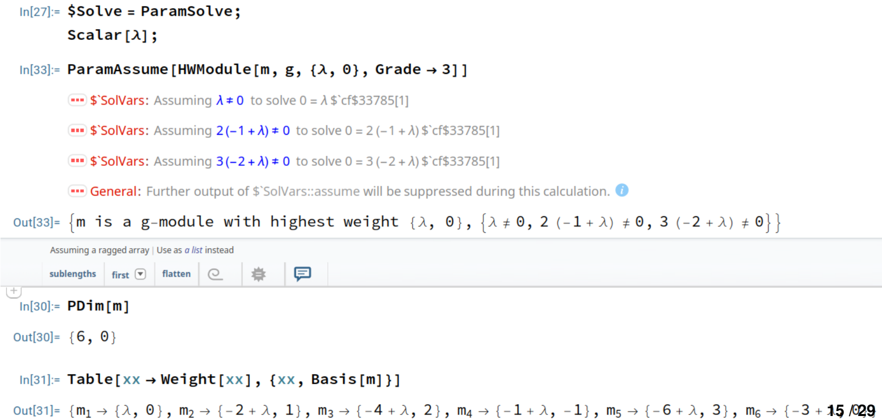
```
In[9]:= Table[xx → Weight[xx], {xx, Basis[v]}]

Out[9]= {v1 → {1, 0}, v2 → {-1, 1}, v3 → {0, -1}}
```

Verma modules



```
In[17]:= HWModule[m, g, {-1, 0}, Grade → 3]

Out[17]= m is a g-module with highest weight {-1, 0}

In[18]:= PDim[m]

Out[18]= {6, 0}

In[20]:= Table[xx → Weight[xx], {xx, Basis[m]}]

Out[20]= {m₁ → {-1, 0}, m₂ → {-3, 1}, m₃ → {-5, 2}, m₄ → {-2, -1}, m₅ → {-7, 3}, m₆ → {-4, 0}}
```

with a parameter λ

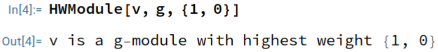


```
In[27]:= $Solve = ParamSolve;
         Scalar[λ];

In[33]:= ParamAssume[HWModule[m, g, {λ, 0}, Grade → 3]]

         $`SolVars: Assuming λ ≠ 0 to solve 0 = λ $`cf$33785[1]

         $`SolVars: Assuming 2(-1 + λ) ≠ 0 to solve 0 = 2 (-1 + λ) $`cf$33785[1]

         $`SolVars: Assuming 3(-2 + λ) ≠ 0 to solve 0 = 3 (-2 + λ) $`cf$33785[1]

         General: Further output of $`SolVars::assume will be suppressed during this calculation.

Out[33]= {m is a g-module with highest weight {λ, 0}, {λ ≠ 0, 2 (-1 + λ) ≠ 0, 3 (-2 + λ) ≠ 0}}
```

Assuming a ragged array | Use as *a list* instead

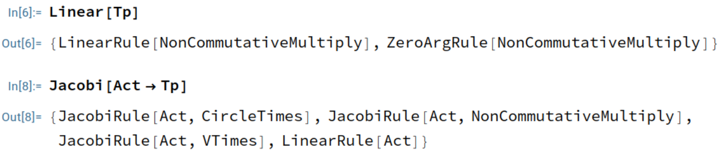
```
In[30]:= PDim[m]

Out[30]= {6, 0}

In[31]:= Table[xx → Weight[xx], {xx, Basis[m]}]

Out[31]= {m₁ → {λ, 0}, m₂ → {-2 + λ, 1}, m₃ → {-4 + λ, 2}, m₄ → {-1 + λ, -1}, m₅ → {-6 + λ, 3}, m₆ → {-3 +
```

Tensor product of modules: $\mathfrak{g} = \mathfrak{sl}(3)$, $V = V_{\pi_1}$



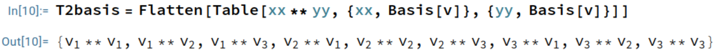
```
In[4]:= HWModule[v, g, {1, 0}]

Out[4]= v is a g-module with highest weight {1, 0}
```

Preparations



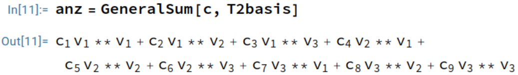
```
In[6]:= Linear[Tp]

Out[6]= {LinearRule[NonCommutativeMultiply], ZeroArgRule[NonCommutativeMultiply]}

In[8]:= Jacobi[Act → Tp]

Out[8]= {JacobiRule[Act, CircleTimes], JacobiRule[Act, NonCommutativeMultiply],
         JacobiRule[Act, VTimes], LinearRule[Act]}
```

Basis in $V \otimes V$



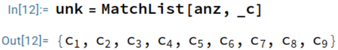
```
In[10]:= T2basis = Flatten[Table[xx ** yy, {xx, Basis[v]}, {yy, Basis[v]}]]

Out[10]= {v1 ** v1, v1 ** v2, v1 ** v3, v2 ** v1, v2 ** v2, v2 ** v3, v3 ** v1, v3 ** v2, v3 ** v3}
```

“Ansatz”



```
In[11]:= anz = GeneralSum[c, T2basis]

Out[11]= c1 v1 ** v1 + c2 v1 ** v2 + c3 v1 ** v3 + c4 v2 ** v1 +
          c5 v2 ** v2 + c6 v2 ** v3 + c7 v3 ** v1 + c8 v3 ** v2 + c9 v3 ** v3
```

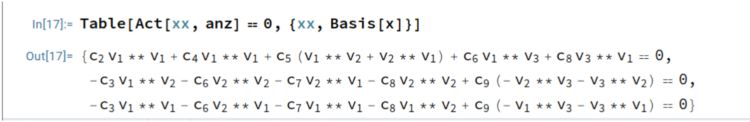
Unknowns



```
In[12]:= unk = MatchList[anz, _c]

Out[12]= {c1, c2, c3, c4, c5, c6, c7, c8, c9}
```

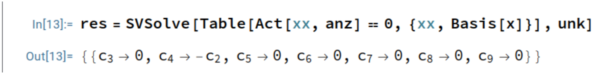
Tensor product of modules: $\mathfrak{g} = \mathfrak{sl}_3$, $V = V_{\pi_1}$

The equations



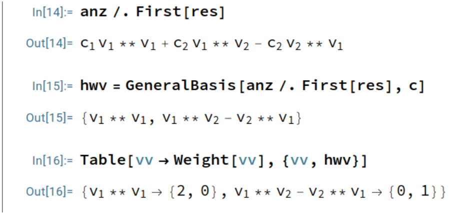
```
In[17]:= Table[Act[xx, anz] == 0, {xx, Basis[x]}]

Out[17]= {c2 v1 ** v1 + c4 v1 ** v1 + c5 (v1 ** v2 + v2 ** v1) + c6 v1 ** v3 + c8 v3 ** v1 == 0,
          -c3 v1 ** v2 - c6 v2 ** v2 - c7 v2 ** v1 - c8 v2 ** v2 + c9 (-v2 ** v3 - v3 ** v2) == 0,
          -c3 v1 ** v1 - c6 v2 ** v1 - c7 v1 ** v1 - c8 v1 ** v2 + c9 (-v1 ** v3 - v3 ** v1) == 0}
```

The highest weight vectors



```
In[13]:= res = SVSolve[Table[Act[xx, anz] == 0, {xx, Basis[x]}], unk]

Out[13]= {{c3 → 0, c4 → -c2, c5 → 0, c6 → 0, c7 → 0, c8 → 0, c9 → 0}}
```

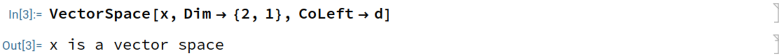
The weights of the highest weight vectors



```
In[14]:= anz /. First[res]

Out[14]= c1 v1 ** v1 + c2 v1 ** v2 - c2 v2 ** v1

In[15]:= hwv = GeneralBasis[anz /. First[res], c]

Out[15]= {v1 ** v1, v1 ** v2 - v2 ** v1}

In[16]:= Table[vv → Weight[vv], {vv, hwv}]

Out[16]= {v1 ** v1 → {2, 0}, v1 ** v2 - v2 ** v1 → {0, 1}}
```

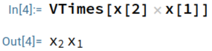
Vectorial Lie (super)algebras

Construct $2|1$-dimensional vector space X with $D = X^*$ (left even linear forms on V)



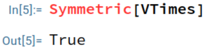
```
In[3]:= VectorSpace[x, Dim → {2, 1}, CoLeft → d]

Out[3]= x is a vector space
```

By default, the multiplication of vectors is free (no relations)

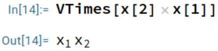


```
In[4]:= VTimes[x[2] × x[1]]

Out[4]= x2 x1
```

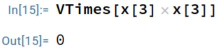
We define the multiplication to be (super)commutative



```
In[5]:= Symmetric[VTimes]

Out[5]= True
```



```
In[14]:= VTimes[x[2] × x[1]]

Out[14]= x1 x2
```



```
In[15]:= VTimes[x[3] × x[3]]

Out[15]= 0
```

and tensor product to be $\mathbb{C}$-linear

```
In[7]:= Linear[Tp];
```

Vectorial Lie (super)algebras

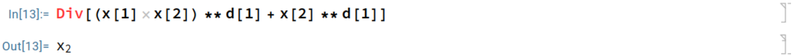
The Lie superalgebra $\mathfrak{vect}(2|1) = \mathrm{Der}\,\mathbb{C}[x_1, x_2|x_3]$

```
In[8]:= VectorLieAlgebra[vec, x]

Out[8]= vec = vect(x)
```

The Lie bracket of vector fields

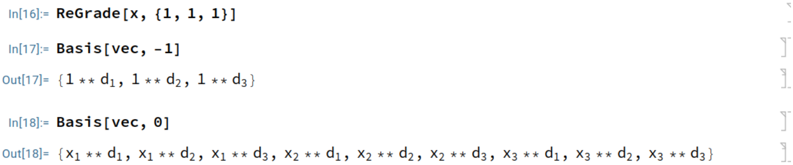
```
In[9]:= Lb[x[1] ** d[1], x[2] ** d[1]]

Out[9]= - x2 ** d1
```

The divergence of a vector field



```
In[13]:= Div[(x[1] × x[2]) ** d[1] + x[2] ** d[1]]

Out[13]= x2
```

The standard $\mathbb{Z}$-grading



```
In[16]:= ReGrade[x, {1, 1, 1}]

In[17]:= Basis[vec, -1]

Out[17]= {1 ** d1, 1 ** d2, 1 ** d3}

In[18]:= Basis[vec, 0]

Out[18]= {x1 ** d1, x1 ** d2, x1 ** d3, x2 ** d1, x2 ** d2, x2 ** d3, x3 ** d1, x3 ** d2, x3 ** d3}
```

Non standard $\mathbb{Z}$-gradings

```
In[19]:= ReGrade[x, {2, 1, 1}]

In[20]:= Basis[vec, -1]

Out[20]= {1 ** d2, 1 ** d3}
```

Poisson Lie (super)algebras

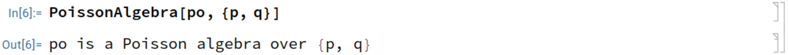
Let P and Q be (super)vector spaces of the same dimension. Let p_i be a basis in P and q_i be a basis in Q such that $\mathsf{p}(p_i) = \mathsf{p}(q_i)$. Then the Poisson bracket it defined by

$$\{f, g\}_{P.b.} = \sum_i (-1)^{\mathsf{p}(f)\mathsf{p}(g)} \left(\sum \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - (-1)^{\mathsf{p}(p_i)} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

Construct vector spaces: $P = Q = \mathbb{C}^{1|1}$

```
In[3]:= VectorSpace[p, Dim → {1, 1}]
        VectorSpace[q, Dim → {1, 1}]
        Symmetric[VTimes]
```

Construct the Poisson algebra $\mathfrak{po}(2|2)$



```
In[6]:= PoissonAlgebra[po, {p, q}]

Out[6]= po is a Poisson algebra over {p, q}
```

Compute the Poisson bracket

```
In[9]:= Pb[p[1] × q[1], p[1] × q[2]]

Out[9]= - p1 q2
```

Contact Lie (super)algebras

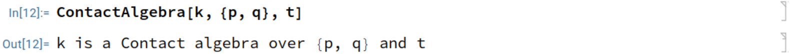
Consider the polynomial algebra over p_i, q_i (as before), and t (even). Then the contact bracket is defined by

$$\{f, g\}_{K.b.} = (2 - E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)(g) + \{f, g\}_{P.b.},$$

where $E = \sum_i p_i \frac{\partial}{\partial p_i} + \sum_i q_i \frac{\partial}{\partial q_I} + \frac{\partial}{\partial t}$.
Construct 1-dimensional space spanned by t

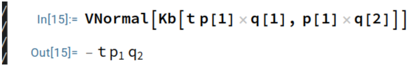
```
In[10]:= TrivialSpace[t]

Out[10]= t
```

Construct the contact Lie supealgebra $\mathfrak{k}(3|2)$



```
In[12]:= ContactAlgebra[k, {p, q}, t]

Out[12]= k is a Contact algebra over {p, q} and t
```

Compute the contact bracket



```
In[15]:= VNormal[Kb[t p[1] × q[1], p[1] × q[2]]]

Out[15]= - t p1 q2
```

Buttin algebra $\mathfrak{b}(n)$

Consider the polynomial algebra over p_i (even), xi_i (odd), $i = 1, \dots, n$. Then the Schouten/anti/Buttin bracket is defined by

$$\{f, g\}_{B.b.} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial \xi_i} + (-1)^{\mathsf{p}(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial p_i}$$

Construct the vector spaces

```
In[20]:= VectorSpace[p, Dim → {2, 0}]
         VectorSpace[ξ, Dim → {0, 2}]
```

Construct $\mathfrak{b}(2)$

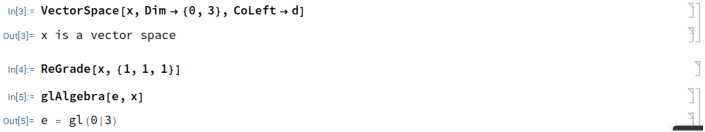
```
In[22]:= ButtinAlgebra[b, {p, ξ}]

Out[22]= b is a Buttin algebra over {p, ξ}
```

Compute the bracket

```
In[24]:= Bb[p[1] × p[2], ξ[1]]

Out[24]= p2
```

Cartan Prolongations

Recall that $\mathfrak{svect}(n|m) = (V, \mathfrak{sl}(n|m))_*$.
We start with $0|3$-dimensional vector space X and $\mathfrak{g} = \mathfrak{sl}(V)$.

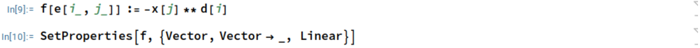


```
In[3]:= VectorSpace[x, Dim → {0, 3}, CoLeft → d]
Out[3]= x is a vector space

In[4]:= ReGrade[x, {1, 1, 1}]

In[5]:= glAlgebra[e, x]
Out[5]= e = gl(0|3)
```

Construct Lie superalgebra $\mathfrak{vect}(V)$

```
In[6]:= Symmetric[VTimes];
        Linear[Tp];

In[8]:= VectorLieAlgebra[vect, x]
Out[8]= vect = vect(x)
```

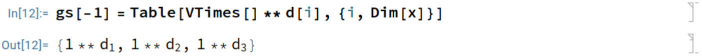
We represent basis elements v_i of V as ∂_i and construct a Lie superalgebras homomorphis $f\colon \mathfrak{gl}(V) \to \mathfrak{vect}_0(V)$ which is given by $f(e_{i,j}) = -x_j \partial_i$



```
In[9]:= f[e[i_, j_]] := -x[j] ** d[i]

In[10]:= SetProperties[f, {Vector, Vector → _, Linear}]
```

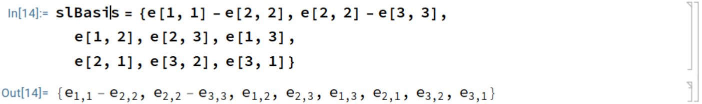
Cartan Prolongations

We construct $\mathfrak{g}_{-1} = \{\partial_1, \partial_2, \partial_3\}$:



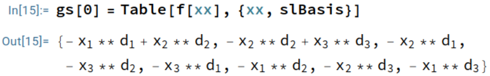
```
In[12]:= gs[-1] = Table[VTimes[] ** d[i], {i, Dim[x]}]

Out[12]= {1 ** d1, 1 ** d2, 1 ** d3}
```

The basis of $\mathfrak{sl}(3)$



```
In[14]:= slBasis = {e[1, 1] - e[2, 2], e[2, 2] - e[3, 3],
           e[1, 2], e[2, 3], e[1, 3],
           e[2, 1], e[3, 2], e[3, 1]}

Out[14]= {e1,1 - e2,2, e2,2 - e3,3, e1,2, e2,3, e1,3, e2,1, e3,2, e3,1}
```

Construct $\mathfrak{g}_0$



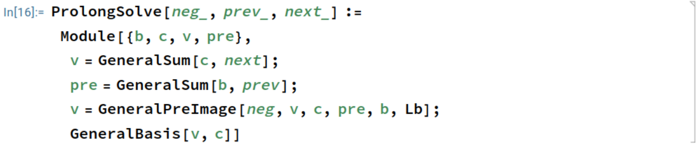
```
In[15]:= gs[0] = Table[f[xx], {xx, slBasis}]

Out[15]= {- x1 ** d1 + x2 ** d2, - x2 ** d2 + x3 ** d3, - x2 ** d1,
          - x3 ** d2, - x3 ** d1, - x1 ** d2, - x2 ** d3, - x1 ** d3}
```

The equation for the prolongation

$$\mathfrak{g}_k = \{X \in \mathfrak{vect}_k(V) \mid [X, \mathfrak{g}_{-1}] \in \mathfrak{g}_{k-1}\}$$

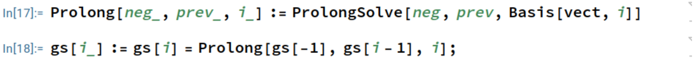
Cartan Prolongations

$$\mathfrak{g}_k = \{X \in \mathfrak{vect}_k(V) \mid [X, \mathfrak{g}_{-1}] \in \mathfrak{g}_{k-1}\}$$



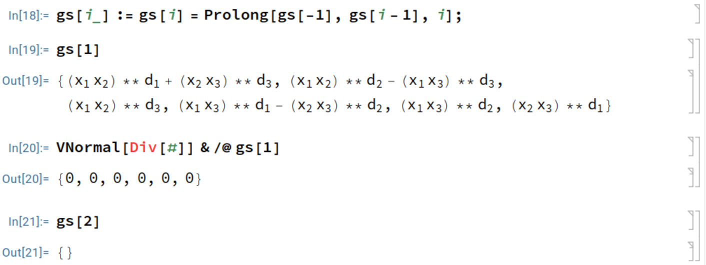
```
In[16]:= ProlongSolve[neg_, prev_, next_] :=
          Module[{b, c, v, pre},
           v = GeneralSum[c, next];
           pre = GeneralSum[b, prev];
           v = GeneralPreImage[neg, v, c, pre, b, Lb];
           GeneralBasis[v, c]]
```

The function to construct $\mathfrak{g}_i$



```
In[17]:= Prolong[neg_, prev_, i_] := ProlongSolve[neg, prev, Basis[vect, i]]

In[18]:= gs[i_] := gs[i] = Prolong[gs[-1], gs[i - 1], i];
```

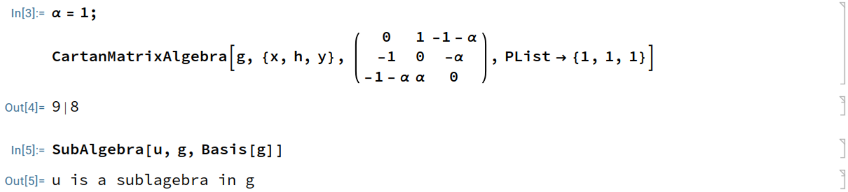
We compute



```
In[18]:= gs[i_] := gs[i] = Prolong[gs[-1], gs[i - 1], i];

In[19]:= gs[1]

Out[19]= {(x1 x2) ** d1 + (x2 x3) ** d3, (x1 x2) ** d2 - (x1 x3) ** d3,
          (x1 x2) ** d3, (x1 x3) ** d1 - (x2 x3) ** d2, (x1 x3) ** d2, (x2 x3) ** d1}

In[20]:= VNormal[Div[#]] & /@ gs[1]

Out[20]= {0, 0, 0, 0, 0, 0}

In[21]:= gs[2]

Out[21]= {}
```

Lie algebra cohomology: $H^2(\mathfrak{g};\mathfrak{g})$, $\mathfrak{g} = \mathfrak{osp}(4|2;1)$

We construct $\mathfrak{osp}(4|2;1)$



```
In[3]:= α = 1;
        CartanMatrixAlgebra[g, {x, h, y}, ( 0       1  -1 - α
                                            -1      0    -α
                                            -1 - α  α    0   ), PList → {1, 1, 1}]

Out[4]= 9|8

In[5]:= SubAlgebra[u, g, Basis[g]]

Out[5]= u is a sublagebra in g
```

Preparations

```
In[6]:= Needs["SuperLie`Cohom`"]

In[7]:= Linear[Tp];
        Jacobi[Act → {wedge, Tp}];
```

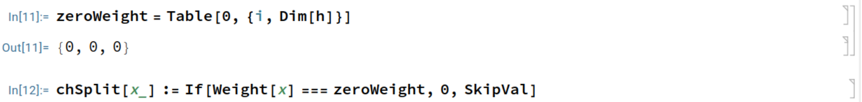
Setup: $H(\mathfrak{u};\mathfrak{u})^\bullet$ with the (trivial) action of $\mathfrak{u}$

```
In[9]:= chSetAlg[u, du, u, u];
        chScalars[b, c]
```

(Note that here “du” is $\Pi\mathfrak{u}^*$.)

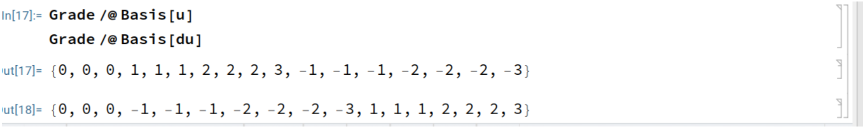
Lie algebra cohomology: $H^2(\mathfrak{u};\mathfrak{u})$, $\mathfrak{g} = \mathfrak{osp}(4|2;1)$

We can skip all cochaing with non-zero weight



```
In[11]:= zeroWeight = Table[0, {i, Dim[h]}]

Out[11]= {0, 0, 0}

In[12]:= chSplit[x_] := If[Weight[x] === zeroWeight, 0, SkipVal]
```

The standard $\mathbb{Z}$-grading on $\mathfrak{u}$ induces a $\mathbb{Z}$-grading on $H^2(\mathfrak{u};\mathfrak{u}) = \oplus_i H^2_i(\mathfrak{u};\mathfrak{u})$.

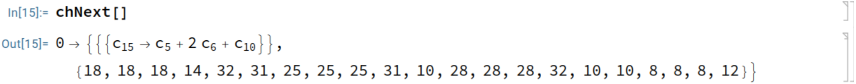


```
In[17]:= Grade /@ Basis[u]
         Grade /@ Basis[du]

ut[17]= {0, 0, 0, 1, 1, 1, 2, 2, 2, 3, -1, -1, -1, -2, -2, -2, -3}

ut[18]= {0, 0, 0, -1, -1, -1, -2, -2, -2, -3, 1, 1, 1, 2, 2, 2, 3}
```

We compute $H^2_0(\mathfrak{u};\mathfrak{u})$

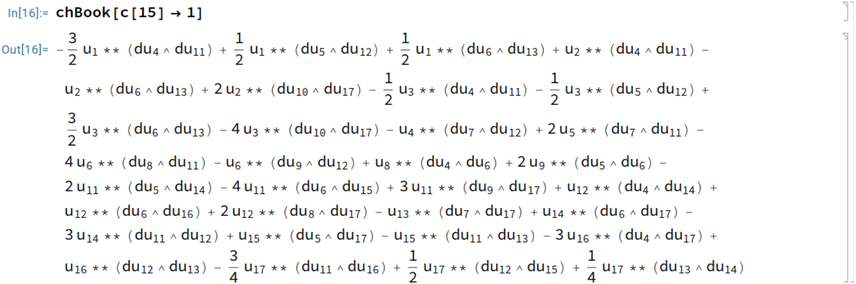
```
In[14]:= chCalc[0, 2]

         Total: {{0, 3}, {3, 20}, {21, 87}}
```

More details



```
In[15]:= chNext[]

Out[15]= 0 → {{{c15 → c5 + 2 c6 + c10}},
           {18, 18, 18, 14, 32, 31, 25, 25, 25, 31, 10, 28, 28, 28, 32, 10, 10, 8, 8, 8, 12}}
```

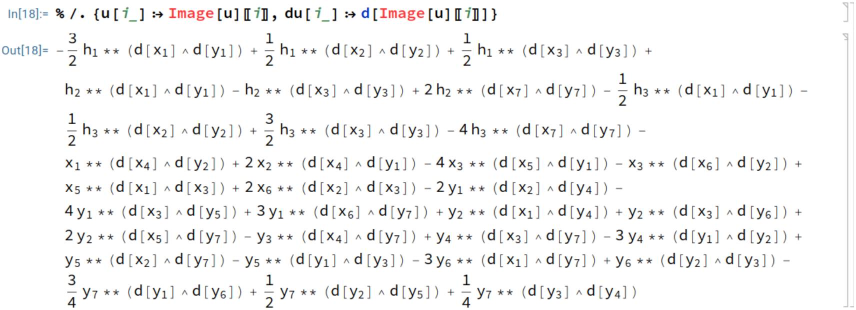
The explicit cocycle



```
In[16]:= chBook[c[15] → 1]
```

$$\text{Out[16]=}\ -\frac{3}{2} u_1 ** (du_4 \wedge du_{11}) + \frac{1}{2} u_1 ** (du_5 \wedge du_{12}) + \frac{1}{2} u_1 ** (du_6 \wedge du_{13}) + u_2 ** (du_4 \wedge du_{11}) - u_2 ** (du_6 \wedge du_{13}) + 2 u_2 ** (du_{10} \wedge du_{17}) - \frac{1}{2} u_3 ** (du_4 \wedge du_{11}) - \frac{1}{2} u_3 ** (du_5 \wedge du_{12}) + \frac{3}{2} u_3 ** (du_6 \wedge du_{13}) - 4 u_3 ** (du_{10} \wedge du_{17}) - u_4 ** (du_7 \wedge du_{12}) + 2 u_5 ** (du_7 \wedge du_{11}) - 4 u_6 ** (du_8 \wedge du_{11}) - u_6 ** (du_9 \wedge du_{12}) + u_8 ** (du_4 \wedge du_6) + 2 u_9 ** (du_5 \wedge du_6) - 2 u_{11} ** (du_5 \wedge du_{14}) - 4 u_{11} ** (du_6 \wedge du_{15}) + 3 u_{11} ** (du_9 \wedge du_{17}) + u_{12} ** (du_4 \wedge du_{14}) + u_{12} ** (du_6 \wedge du_{16}) + 2 u_{12} ** (du_8 \wedge du_{17}) - u_{13} ** (du_7 \wedge du_{17}) + u_{14} ** (du_6 \wedge du_{17}) - 3 u_{14} ** (du_{11} \wedge du_{12}) + u_{15} ** (du_5 \wedge du_{17}) - u_{15} ** (du_{11} \wedge du_{13}) - 3 u_{16} ** (du_4 \wedge du_{17}) + u_{16} ** (du_{12} \wedge du_{13}) - \frac{3}{4} u_{17} ** (du_{11} \wedge du_{16}) + \frac{1}{2} u_{17} ** (du_{12} \wedge du_{15}) + \frac{1}{4} u_{17} ** (du_{13} \wedge du_{14})$$

In terms of Chevalley basis



```
In[18]:= % /. {u[i_] ⧴ Image[u]⟦i⟧, du[i_] ⧴ d[Image[u]⟦i⟧]}
```

$$\text{Out[18]=}\ -\frac{3}{2} h_1 ** (d[x_1] \wedge d[y_1]) + \frac{1}{2} h_1 ** (d[x_2] \wedge d[y_2]) + \frac{1}{2} h_1 ** (d[x_3] \wedge d[y_3]) + h_2 ** (d[x_1] \wedge d[y_1]) - h_2 ** (d[x_3] \wedge d[y_3]) + 2 h_2 ** (d[x_7] \wedge d[y_7]) - \frac{1}{2} h_3 ** (d[x_1] \wedge d[y_1]) - \frac{1}{2} h_3 ** (d[x_2] \wedge d[y_2]) + \frac{3}{2} h_3 ** (d[x_3] \wedge d[y_3]) - 4 h_3 ** (d[x_7] \wedge d[y_7]) - x_1 ** (d[x_4] \wedge d[y_2]) + 2 x_2 ** (d[x_4] \wedge d[y_1]) - 4 x_3 ** (d[x_5] \wedge d[y_1]) - x_3 ** (d[x_6] \wedge d[y_2]) + x_5 ** (d[x_1] \wedge d[x_3]) + 2 x_6 ** (d[x_2] \wedge d[x_3]) - 2 y_1 ** (d[x_2] \wedge d[y_4]) - 4 y_1 ** (d[x_3] \wedge d[y_5]) + 3 y_1 ** (d[x_6] \wedge d[y_7]) + y_2 ** (d[x_1] \wedge d[y_4]) + y_2 ** (d[x_3] \wedge d[y_6]) + 2 y_2 ** (d[x_5] \wedge d[y_7]) - y_3 ** (d[x_4] \wedge d[y_7]) + y_4 ** (d[x_3] \wedge d[y_7]) - 3 y_4 ** (d[y_1] \wedge d[y_2]) + y_5 ** (d[x_2] \wedge d[y_7]) - y_5 ** (d[y_1] \wedge d[y_3]) - 3 y_6 ** (d[x_1] \wedge d[y_7]) + y_6 ** (d[y_2] \wedge d[y_3]) - \frac{3}{4} y_7 ** (d[y_1] \wedge d[y_6]) + \frac{1}{2} y_7 ** (d[y_2] \wedge d[y_5]) + \frac{1}{4} y_7 ** (d[y_3] \wedge d[y_4])$$

Thank you