Primitive forms for symplectic twistor operators

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1) Symplectic manifolds and connections

Let (M, ω) be a symplectic manifold, i.e., ω point-wise non-degenerate antisymmetric differential 2-form and $d\omega = 0$.

Let ∇ be a symplectic ($\nabla\omega = 0$) and a torsion-free connection. Non-unique in contrary to Riemannian geometry. Form affine space modeled on smooth sections $\Gamma(Sym^3(T^*M))$ of the bundle $Sym^3(T^*M)$ (Libermann; Tondeur)

Symplectic manifolds with a symplectic torsion-free connections - Fedosov manifolds
They are used for *deformation quantization* (of Poisson algebra of smooth functions on M, an L_∞-morphism)

Examples: Kähler manifolds, bilagrangian and bipolarized manifolds with the so called canonical connection

Symplectic spinors - algebraic model

(V, ω_0) real symplectic vector space of dimension $2n$, G the symmetry group, i.e., the symplectic group $Sp(V, \omega_0)$

$\lambda : \widetilde{G} \to G$, connected Lie group double cover of G; $\widetilde{G}$ - the so called **metaplectic group**, non-matrix Lie group, but also $2:1$ covering as $Spin(m) \to SO(m)$ known from the Euclidean case

Representation of $Mp(V, \omega_0)$
$U \subseteq V$ maximal isotropic vector subspace: $\omega(v, w) = 0$ for all $v, w \in U$, $U \simeq \mathbb{R}^n$. Take $S = L^2(U)$ w.r. to a scalar product on U

Symplectic spinors - properties

Let $L : \widetilde{G} \to \mathcal{U}(L^2(U))$ be the so called symplectic spinor representation
Unitary (= into $\mathcal{U}$ (of S)), faith-full (= L is injective).
Its Harish-Chandra module (= 'infinitesimal structure') is $\bigoplus_{i=0}^{\infty} Sym^i(U) \simeq Pol(x_1, \ldots, x_n)$.

Also known as Segal–Shale–Weil, metaplectic, oscillator representation: [Shale], [Weil].

Discovered by quantization of Klein–Gordon fields (Shale/Segal), symmetry of ϑ-functions (Weil)

2 Model for the symplectic twistor complex

$E^i = \bigwedge^i V^* \otimes S$ - symplectic spinor-valued wedge i-forms
$E = \bigoplus_{i=0}^{2n} \bigwedge^i V^* \otimes S$ - **symplectic spinor-valued wedge forms**

$$\rho(g)(\alpha \otimes s) = \lambda(g)^* \alpha \otimes L(g)s, \alpha \otimes s \in E^i, g \in \widetilde{G}$$

Remark: Model for Dolbeault complex: $U(T_x M, J_x, g_x)$-module $\bigoplus_{0 \leq p+q \leq 2n} \bigwedge^p (T_x M^{1,0})^* \otimes \bigwedge^q (T_x M^{0,1})^*$, $(T_x M, J_x, g_x)$ hermit. vect. space

Thm. [KrLie]: The module E decomposes as $\widetilde{G}$-module into direct sum

$$\bigoplus_{(i,j) \in K} E^{ij}, \text{ where } K \text{ is a set with } (n+1)^2 \text{ elements,}$$

$E^{ij} = E^{ij,+} \oplus E^{ij,-} \subseteq E^i$ and $E^{ij,\pm}$ **are irreducible** $\widetilde{G}$-modules.

Decomposition of symplectic spinor-valued wedge forms

Dim $M = 6$

$\mathbf{E}^0$	$\mathbf{E}^1$	$\mathbf{E}^2$	$\mathbf{E}^3$	$\mathbf{E}^4$	$\mathbf{E}^5$	$\mathbf{E}^6$
E^{00}	E^{10}	E^{20}	E^{30}	E^{40}	E^{50}	E^{60}
	E^{11}	E^{21}	E^{31}	E^{41}	E^{51}	
		E^{22}	E^{32}	E^{42}		
			E^{33}			

Curvature tensor of symplectic connections

Components of curvature tensor R of a symplectic torsion-free connection known (Vaisman [Vais]):

(1) No intrinsic scalar curvature.
(2) Only symplectic Ricci σ and symplectic Weyl components.
(3) $R = \Sigma + W$, Σ - constructed using σ and ω only

Definition: Symplectic torsion-free connection ∇ is called **symplectic Weyl-flat** if the Weyl tensor vanishes.

In [Vais] *symplectic curvature tensor reducible*.

Examples of Weyl-flat symplectic manifolds

[1] Bilagrangian and bipolarized structures. Need not be Kähler. Conditions for flat and Ricci-flat, see [Vais].

[2] Easy to derive: Complex Kähler is Weyl-flat if its holomorphic sectional curvature is constant

Theorem (Hawley '53, Igusa '54): If the holomorphic sectional curvature of a geodesically complete Kähler manifold is constant, it is covered symplectomorphically by CP^n, C^n or B^n (open ball) with their canonical complex and classical riemannian structures.

[3] *Kodaira–Thurston manifold* with a specific symplectic connection which is flat ([Fox]).

Transferring the model to the metaplectic structures

If a symplectic manifold (M, ω) admits a symplectic spin structure ($=$ symplectic analogue of the riemannian or pseudoriemannian spin structure), denoted by $\mathcal{P} \implies$

Form associated bundles $\mathcal{E} = \mathcal{P} \times_\rho E$ - bundle of Hilbert spaces on M

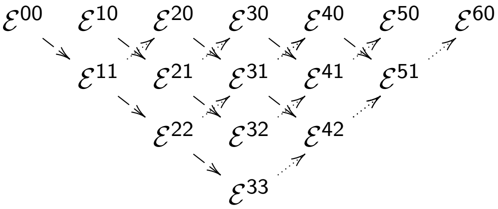
Form also $\mathcal{E}^{ij} = \mathcal{P} \times_\rho E^{ij}$

For a symplectic connection, form covariant derivatives $d^\nabla : \Gamma(\mathcal{E}^i) \to \Gamma(\mathcal{E}^{i+1})$

metaplectic structure $=$ a two-fold covering of the bundle of symplectic frames on (M, ω) equipped with an action of the metaplectic group, see Suppl. below

Sequences of symplectic twistor operators

Definition: Let ∇ be a connection preserving the symplectic form on a symplectic manifold (M, ω). Then $T_{\pm}^{ij} = p^{i+1, j\pm 1} d^{\nabla}_{|\Gamma(\mathcal{E}^{ij})}$ is called the $(\pm)$-**symplectic twistor operator**.
Dim $M = 4$



Complexes for (M, ω, ∇) with a metaplectic structure

Theorem: If ∇ is symplectic, torsion-free and Weyl flat, then for any i, j, the sequence $\left(\Gamma(E^{i+k,j\pm k}), T_{\pm}^{i+k,j\pm k}\right)_{k\in\mathbb{Z}}$ is a complex, i.e., $T_{\pm}^{i+k+1,j\pm k\pm 1} T_{\pm}^{i+k,j\pm k} = 0$.

Proof. [KryCliffAlg].

Definition: Symplectic spinor-valued form $\phi = \alpha \otimes s \in \mathcal{E}^i$ is called **primitive** if it is an element in the kernel of the map

$$Y(\alpha \otimes s) = \sum_{i,j=1}^{2n} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s,$$

where α is differential form, s is symplectic spinor field, ι denotes insertion, and $\cdot$ so-called symplectic spinor multiplication.

Lemma(rep-thy-lemma): Symplectic spinor-valued form is primitive if and only if it is a section of $\mathcal{E}^{ii}$.
Proof. Follows from [KrLie] immediately.

Lefschetz decomposition

Set $X(\alpha \otimes s) = \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes e_i \cdot s$, where α is a differential form and s is a symplectic spinor field

Theorem (Lefschetz type decomp.): For (M, ω, ∇) a symplectic manifold with Weyl-type connection

$$\Gamma(\mathcal{E}^i) = \bigoplus_{j=0}^{i} X^{i-j}\Gamma(\mathcal{E}^{jj})$$

Proof. Lemma, $\widetilde{G}$-equivariance of X and decomposition structure E.

4) Lefschetz-type map [X]

Definition: The (+)-**primitive cohomology group** is the quotient

$$H_T^{i,j}(M) = \mathrm{Ker}\, T_+^{i,j} / \mathrm{Im}\, T_+^{i-1,j-1}.$$

The $+$-case is for simplicity.

Lefschetz type map

Theorem: If (M, ω, ∇) is a symplectic manifopld with connection of Weyl-type, then X descends to the twistor cohomology groups, i.e., $[X] : H_T^{i,j}(M) \to H_T^{i+1,j}(M)$, $[X][\phi] := [X(\phi)]$, is a well defined linear map.

Proof. $[\psi] = 0 \implies \psi \in \operatorname{Im} T_+^{i-1,j-1} \implies \psi = p^{i,j} d^\nabla \phi \implies$
$X\psi = Xp^{i,j} d^\nabla \phi$.
By Schur lemma for intertwining operators: $Xp^{i,j} = -\lambda p^{i+1,j} X$ for a constant λ, possibly zero.
Thus $X\psi = -\lambda p^{i+1,j} X d^\nabla \phi$. It is easy to compute that
$Xd^\nabla = -d^\nabla X$ using the torsion-free property.
To sum-up $X\psi = -\lambda p^{i+1,j} X d^\nabla \phi = p^{i+1,j} d^\nabla X(\lambda\phi) = T_+^{i,j-1}(\lambda\phi)$, thus it is in the image of $T_+^{i,j-1}$. $\square$

Suppl.: Definition of metaplectic structure

(M, ω) symplectic manifold
$\mathcal{Q} = \{A : V \to T_m M \,|\, \omega_0(u, v) = \omega_m(Au, Av), u, v \in V, m \in M\}$ is a principal G-bundle, bundle of symplectic frames, $\pi_Q : \mathcal{Q} \to M$
If $\pi_P : \mathcal{P} \to M$ is principal $\widetilde{G}$-bundle and $\Lambda : \mathcal{P} \to \mathcal{Q}$ is a fibre bundle map, $(\mathcal{P}, \Lambda)$ is called **metaplectic structure** on (M, ω) if the diagram

$$\begin{array}{ccccc} P \times \widetilde{G} & \longrightarrow & P & & \\ \big\downarrow {\scriptstyle \Lambda \times \lambda} & & \big\downarrow {\scriptstyle \Lambda} & \searrow^{\pi_P} & \\ & & & & M \\ & & & \nearrow_{\pi_Q} & \\ Q \times G & \longrightarrow & Q & & \end{array}$$

commutes.
Thm. (Forger, Hess): A metaplectic structure exists iff $c_1(TM^c)$ is even, i.e. an element of $H^2(M, 2\mathbb{Z})$ iff $w_1(TM) = 0$.

Suppl.: Ellipticity of the subcomplexes

Theorem: If ∇ is symplectic, torsion-free and Weyl flat, then

$$\left(\Gamma(\mathcal{E}^{i+k,k}),\, T^{i+k,k,+}\right)_{-1\leq k\leq \lfloor \frac{2n-i}{2} \rfloor}$$

$i = 0, \ldots, 2n-2$, and

$$\left(\Gamma(\mathcal{E}_{\lfloor \frac{i+1}{2} \rfloor + k, \lfloor \frac{i}{2} \rfloor - k}),\, T^{\lfloor \frac{i+1}{2} \rfloor + k, \lfloor \frac{i}{2} \rfloor - k, -}\right)_{0\leq k\leq \lfloor \frac{i}{2} \rfloor + 1}$$

are elliptic for $i = 2, \ldots, 2n$.

[CMP79] Forger, M., Hess, H., Universal metaplectic structures and geometric quantization, Commun. Math. Phys. 67 (1979), 267–278.

[Hab] Habermann, K., The Dirac operator on symplectic spinors, Ann. Global Anal. Geom. 13 (1995), no. 2, 155–168.

[Habs] Habermann, K., Habermann, L., Introduction to symplectic Dirac operators, Lecture Notes in Mathematics 1887. Springer–Verlag, 2006.

[Haw] Hawley, N., Constant holomorphic curvature. Canad. J. Math. 5 (1953), 53-56.

[Ig] Igusa, J., On the structure of a certain class of Kaehler varieties, Amer. J. Math. 76 (1954), 669-678.

[Kos] Kostant, B., Symplectic spinors. Symposia Mathematica, Vol. XIV (Convegno di Geometria Simplettica e Fisica Matematica, INDAM, Rome, 1973), 139–152, Academic Press, London, 1974.

[KrMon] Krýsl, S., Complex of twistor operators in spin symplectic geometry, Monatshefte für Mathematik, Vol. 161 (2010), no.4, 381–398.

[KrLie] Krýsl, S., Howe duality for the metaplectic group acting on symplectic spinor valued forms, Journal of Lie theory, Vol. 22 (2012), no. 4, 1049–1063; arxiv 2008.

[KryCliffAlg] Krýsl, S., Complexes of symplectic twistor operators, Cliff. Anal. Alg., 2022.

[Sh] Shale, D., Linear symmetries of free boson fields, Trans. Amer. Math. Soc. 103 (1962), 149–167.

[Slup] Slupinski, M., A Hodge type decomposition for spinor valued forms, Ann. Sci.École Norm. Sup. (4) 29 (1996), no. 1, 23–48.

[Vais] Vaisman, I., Symplectic curvature tensors, Monatshefte für Mathematik 100 (1985), 299–327.

[Weil] Weil, A., Sur certains groupes d'opérateurs unitaires, Acta Math. 111 (1964), 143–211.