Graded-equivariant bundle gerbe K-theory

Tim Lüders - based on WIP w/ K. Waldorf & L. Otto

University of Vienna

January 23, 2025

Motivation

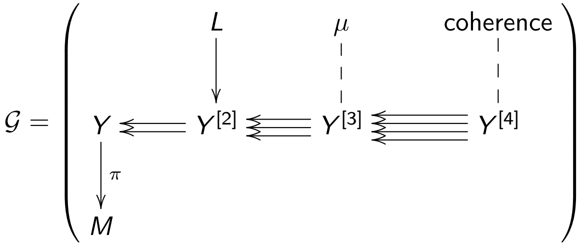
Graded Lie groupoid	Appearence of $\mathrm{Grb}(\Gamma, \phi)$ & $\mathrm{K}^{(\mathcal{G}, \mathcal{Q}, \psi)}(\Gamma, \phi)$
$(M_{\mathrm{dis}}, 1)$	Lifting problems
(BG, ϕ)	Wigner's theorem
$(M /\!\!/ G, \phi)$	Orientifold sigma models
$(G /\!\!/ G, 1)$	Chern-Simons theory
(Γ, ϕ)	Topological insulators
$(\Gamma \rtimes_{\tau} \mathbb{Z}_2, \phi_{\tau})$	String theory?

Overview

- Bundle gerbes
- Twisted K-theory
- Graded Lie groupoids
- Graded-equivariant bundle gerbe K-theory
- Applications and special cases

Bundle gerbes

Simplicial Structure of bundle gerbes:



- $\pi\colon Y \to M$ is a surjective submersion
- L is a complex line bundle over $Y^{[2]} := \{(y, y') \in Y^2 | \pi(y) = \pi(y')\}$
- $\mu\colon \operatorname{pr}_{23}^* L \otimes \operatorname{pr}_{12}^* L \to \operatorname{pr}_{13}^* L$ is an associative isomorphism

Example: $Y = M,\ L = M \times \mathbb{C},\ \mu =$ multiplication. Notation: $\mathcal{I}$

Bundle gerbes

1-morphisms of bundle gerbes $(\zeta, W, \phi)\colon \mathcal{G}_1 \to \mathcal{G}_2$:

- $\zeta\colon Z \to Y_1 \times_M Y_2$ a surjective submersion
- W a complex vector bundle over Z
- an isomorphism of vector bundles

$$\phi\colon \ \mathrm{pr}_2^* W \otimes L_1 \longrightarrow L_2 \otimes \mathrm{pr}_1^* W$$

over $Z^{[2]}$ compatible with μ_1 and μ_2.

2-morphisms $(\rho, \xi)\colon (\zeta, W, \phi) \to (\zeta', W', \phi')$:

- $\rho\colon V \to Z \times_{(Y_1 \times_M Y_2)} Z'$ a surjective submersion
- $\xi\colon W \to W'$ a morphism compatible with ϕ and ϕ'

Bundle gerbes

Observation: Bundle gerbes over M form a bicategory $\mathrm{Grb}(M)$.

Observation: Bundle gerbes *categorify* vector bundles

$$\mathrm{Hom}_{\mathrm{Grb}(M)}(\mathcal{I}, \mathcal{I}) \cong \mathrm{VectBdl}(M)$$

Theorem: *Classification of line bundles and bundle gerbes*

$$\mathrm{h}_0(\mathrm{LineBdl}(M)) \cong H^2(M, \mathbb{Z}), \qquad \mathrm{h}_0(\mathrm{Grb}(M)) \cong H^3(M, \mathbb{Z})$$

induced by the *Chern* and *Dixmier-Douady* class, respectively.

Observation: $\mathrm{Grb}(M)$ is symmetric monoidal, rigid, framed, etc.

Observation: The assignment $\mathrm{Grb}\colon \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{BiCat}$ is a *sheaf*

Twisted K-theory

Fix $\mathcal{G} \in \mathsf{Grb}(M)$.

Definition: The category of $\mathcal{G}$-*twisted vector bundles* over M is

$$\mathsf{VectBdl}^{\mathcal{G}}(M) := \mathsf{Hom}_{\mathsf{Grb}(M)}(\mathcal{G}, \mathcal{I})$$

Observation: $(\mathsf{h}_0(\mathsf{VectBdl}^{\mathcal{G}}(M)), \oplus)$ is a commutative monoid

Definition: The $\mathcal{G}$-*twisted K-group of M* is

$$\mathsf{K}^{\mathcal{G}}(M) := \mathsf{K}(\mathsf{h}_0(\mathsf{VectBdl}^{\mathcal{G}}(M)), \oplus)$$

where the second K is Grothendieck's group completion.

Observation: $\mathsf{K}^{\mathcal{I}}(M) \cong \mathsf{K}(M)$

Graded Lie groupoids

Definition: A *graded Lie groupoid* (Γ, ϕ) is a Lie groupoid Γ together with a smooth functor $\phi\colon \Gamma \to \mathrm{B}\mathbb{Z}_2$.

Examples:

- $(M_{\text{dis}}, 1)$ for M a smooth manifold
- (BG, ϕ) for (G, ϕ) a graded Lie group
- $(M /\!\!/ G, \phi)$ for a graded Lie group (G, ϕ) acting on M
- $(\Gamma, 1)$ for Γ a Lie groupoid
- $(\Gamma \rtimes_\tau \mathbb{Z}_2, \phi_\tau)$ for (Γ, τ) a Real Lie groupoid

Graded-equivariant bundle gerbe K-theory

Question: What is a bundle gerbe over a Lie groupoid Γ?

- Applying the *nerve construction* yields a simplicial manifold

$$\Gamma_\bullet := \left(\Gamma_0 \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} \Gamma_1 \underset{\partial_2}{\overset{\partial_0}{\Lleftarrow}} \Gamma_2 \underset{\partial_3}{\overset{\partial_0}{\Lleftarrow}} \Gamma_3 \dots \right).$$

- Evaluating Grb yields a cosimplicial bicategory $\mathrm{Grb}(\Gamma_\bullet)$.

- Taking a homotopy limit yields the bicategory $\mathrm{Grb}(\Gamma)$

Task: Spell this out, generalize to graded Lie groupoids

Graded-equivariant bundle gerbe K-theory

Question: What is a bundle gerbe over a graded Lie groupoid (Γ, ϕ)?

Answer: A *bundle gerbe over* (Γ, ϕ) consists of:

- a bundle gerbe $\mathcal{G}$ over Γ_0,
- a 1-isomorphism of bundle gerbes

$$\mathcal{Q} : \partial_0^* \mathcal{G} \to \partial_1^* \mathcal{G}^{\phi}$$

  over Γ_1, and
- an associative 2-isomorphism of bundle gerbes

$$\psi : \partial_2^* \mathcal{Q} \otimes \partial_0^* \mathcal{Q}^{\partial_2^* \phi} \to \partial_1^* \mathcal{Q}$$

  over Γ_2.

Here $(-)^{\phi}$ denotes complex conjugation iff $\phi = -1$

Graded-equivariant bundle gerbe K-theory

Bundle gerbes over (Γ, ϕ) form a bicategory $\mathsf{Grb}(\Gamma, \phi)$.

Fix $(\mathcal{G}, \mathcal{Q}, \psi) \in \mathsf{Grb}(\Gamma, \phi)$.

Definition: The category of $(\mathcal{G}, \mathcal{Q}, \psi)$-*twisted vector bundles* over (Γ, ϕ) is

$$\mathsf{VectBdl}^{(\mathcal{G},\mathcal{Q},\psi)}(\Gamma, \phi) := \mathsf{Hom}_{\mathsf{Grb}(\Gamma,\phi)}((\mathcal{G}, \mathcal{Q}, \psi), (\mathcal{I}, \mathsf{id}, \mathsf{id}))$$

Observation: $(\mathsf{h}_0(\mathsf{VectBdl}^{(\mathcal{G},\mathcal{Q},\psi)}(\Gamma, \phi)), \oplus)$ is a commutative monoid

Definition: The $(\mathcal{G}, \mathcal{Q}, \psi)$-*twisted K-group of* (Γ, ϕ) is

$$\mathsf{K}^{(\mathcal{G},\mathcal{Q},\psi)}(\Gamma, \phi) := \mathsf{K}(\mathsf{h}_0(\mathsf{VectBdl}^{(\mathcal{G},\mathcal{Q},\psi)}(\Gamma, \phi), \oplus)$$

where the second K is Grothendieck's group completion.

Applications & Special cases

Graded Lie groupoid	Appearence of $\mathsf{Grb}(\Gamma, \phi)$ & $\mathsf{K}^{(\mathcal{G},\mathcal{Q},\psi)}(\Gamma, \phi)$
$(M_{\mathsf{dis}}, 1)$	Lifting problems
(BG, ϕ)	Wigner's theorem
$(M /\!\!/ G, \phi)$	Orientifold sigma models
$(G /\!\!/ G, 1)$	Chern-Simons theory
(Γ, ϕ)	Topological insulators
$(\Gamma \rtimes_\tau \mathbb{Z}_2, \phi_\tau)$	String theory?