Courant Algebroids

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Outline

- Why?
- What?
- How?
- Where?

Motivation

Algebroids generalize the standard differential geometry and the notion of linear algebras.

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Certain Courant algebroid provides a geometrical playground for the physical theory of supergravity.

Notation I

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- Module of generalized $(p, q)-$tensors on M:

Notation II

Note: Whenever our vector bundle turns out to be the tangent bundle, we shall simplify the notation such as in the following:

$$\mathscr{T}^p_q(M; TM) =: \mathscr{T}^p_q(M)$$

Leibniz Algebra

Leibniz algebra L is a vector space with a bilinear mapping:

$$[\cdot, \cdot] : L \times L \to L$$

such that the Leibniz identity holds, that is:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

for all $a, b, c \in L$.

Leibniz Algebroid

Leibniz algebroid is a vector bundle (E, π, M) along with a vector bundle homomorphism $\rho : E \to TM$ and a bilinear mapping:

$$[\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

such that the Leibniz identity and Leibniz rule in the second input hold, that is:

$$[e, [e', e'']_E]_E = [[e, e']_E, e'']_E + [e', [e, e'']_E]_E,$$

$$[e, fe']_E = f[e, e']_E + (\rho(e) \cdot f)e'.$$

The map ρ is called the anchor of Leibniz algebroid.

Generalized Lie Derivative

Generalized Lie derivative $\mathscr{L}^E$ along the section e of E is linear operator on the graded algebra of generalized tensor fields on M defined in the following way:

$$\mathscr{L}_e^E f = \rho(e) \cdot f, \ \forall f \in C^\infty(M),$$

$$\mathscr{L}_e^E e' = [e, e']_E, \ \forall e' \in \Gamma(E),$$

$$\langle \mathscr{L}_e^E \alpha, e' \rangle = \rho(e) \cdot \langle \alpha, e' \rangle - \langle \alpha, [e, e']_E \rangle, \ \forall \alpha \in \Gamma(E^*), e' \in \Gamma(E),$$

$$[\mathscr{L}_e^E \tau](e_1, \ldots, e_q; \alpha_1, \ldots, \alpha_p) = \rho(e)\tau(e_1, \ldots, e_q; \alpha_1, \ldots, \alpha_p)$$
$$- \tau(\mathscr{L}_e^E e_1, \ldots, e_q; \alpha_1, \ldots, \alpha_p) - \ldots - \tau(e_1, \ldots, e_q; \alpha_1, \ldots, \mathscr{L}_e^E \alpha_p)$$
$$\forall e_1, \ldots, e_q \in \Gamma(E), \alpha_1, \ldots, \alpha_p \in \Gamma(E^*), \tau \in \mathscr{T}_q^p(M; E)$$

Leibniz Algebroids IV

Examples:

- $(TM \xrightarrow{\pi} M, id_{TM}, [\cdot, \cdot])$
- $(L \xrightarrow{\pi} m, 0, [\cdot, \cdot])$
- $(TM \oplus \Lambda^p T^* M \xrightarrow{\pi} M, pr_1, [\cdot, \cdot]_D)$, with the Dorfman bracket $[\cdot, \cdot]_D$ defined as:

$$[X + \xi, Y + \eta]_D := [X, Y] + \mathscr{L}_X \eta - i_Y d\xi$$

$\forall X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^p(M)$

Grupoid

Grupoid consists of two sets G and M with two maps $\alpha, \beta : G \to M$ called source and target, the so-called object inclusion map $1_M : M \to G$ and a partial multiplication function in G such that the following holds:

- $\alpha(hg) = \alpha(g), \; \beta(hg) = \beta(h),$
- $j(hg) = (jh)g,$
- $\alpha(1_x) = \beta(1_x) = x,$
- $g1_{\alpha(g)} = g, \; 1_{\beta(g)}g = g$

$\forall j, h, g \in G$ and $x \in M$ whenever the partial multiplication between such elements is defined. Each element $g \in G$ also has its two-sided inverse $g^{-1} \in G$.

(Lie) Grupoid II

Note: The definition of grupoid can be smoothly rephrased using category theoretical language by simply stating that a grupoid is a category where each morphism is isomorphism.

(Lie) Grupoid III

Lie Grupoid

Lie grupoid (G, M) involves besides the standard grupoid data also a smooth manifold structure on both G and M.

Lie Algebra

Lie algebra is a vector space $\mathscr{G}$ equipped with a bilinear mapping:

$$[\cdot,\cdot] : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$$

such that the Jacobi identity holds and the mapping alternates, that is:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,$$

$$[x,x] = 0,$$

$\forall x, y, z \in \mathscr{G}$.

Lie Algebroid

Lie algebroid is a Leibniz algebroid $(L \xrightarrow{\pi} M, l, [\cdot, \cdot]_L)$, for which the Leibniz bracket alternates.

Note: Another way to define Lie algebroid is to say that it is a vector bundle over the set of units of a Lie grupoid.

Generalized Differential

In the Lie algebroid settings, there is an operator $d_L : \Omega^\bullet(M; L) \to \Omega^{\bullet+1}(M; L)$ on the generalized exterior algebra defined as follows:

$$(d_L\omega)(e_0, e_1, \dots, e_p) := \sum_{i=0}^{p} (-1)^i l(e_i)\omega(e_0, \dots, \hat{e}_i, \dots, e_p) +$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([e_i, e_j]_L, e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_p)$$

$\forall \omega \in \Omega^p(M; L)$ and $e_0, e_1, \dots, e_p \in \Gamma(L)$.

Note: The generalized differential encodes the entire Lie algebroid information.

Examples:

- $(TM \xrightarrow{\pi} M, id_{TM}, [\cdot,\cdot])$,
- $(\mathscr{G} \xrightarrow{\pi} m, 0, [\cdot,\cdot])$,
- $(T^*M \xrightarrow{\pi'} M, \Pi, [\cdot,\cdot]_\Pi)$ with:

$$\Pi(\alpha) \equiv \Pi(\alpha,\cdot) \text{ and } [\alpha,\beta]_\Pi = \mathscr{L}_{\Pi(\alpha,\cdot)}\beta - i_{\Pi(\beta,\cdot)}d\alpha$$

where $\Pi \in \mathfrak{X}^2(M)$.

Fiber-wise Metric

Fiber-wise metric on a vector bundle is a symmetric bilinear non-degenerate form $\langle \cdot, \cdot \rangle_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$.

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Courant Algebroid

Courant algebroid is a pair of Leibniz algebroid and a fiber-wise metric $(E \xrightarrow{\pi} M, \rho, [\cdot, \cdot]_E, \langle \cdot, \cdot \rangle_E)$, such that:

$$\rho(e) \cdot \langle e', e'' \rangle_E = \langle [e, e']_E, e'' \rangle_E + \langle e', [e, e'']_E \rangle_E,$$

$$\langle [e, e]_E, e' \rangle_E = \frac{1}{2} \rho(e') \cdot \langle e, e \rangle_E,$$

$\forall e, e', e'' \in \Gamma(E)$.

Note: Equivalently, we have $\mathscr{L}_e^E g_E = 0$ and $[e, e]_E = \frac{1}{2}\mathscr{D}\langle e, e\rangle_E$ for each $e \in \Gamma(E)$ where g_E is the tensor corresponding to the fiber-wise metric and $\mathscr{D} = g_E^{-1} \circ \rho^T \circ d : C^\infty(M) \to \Gamma(E)$.

Examples:

- $(\mathscr{G} \xrightarrow{\pi} m, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$, for $\mathscr{G}$ a quadratic Lie algebra
- $(TM \oplus T^*M \xrightarrow{\pi} M, pr_1, [\cdot,\cdot]_D, \langle\cdot,\cdot\rangle)$, for $[\cdot,\cdot]_D$ a Dorfman bracket and $\langle\cdot,\cdot\rangle$ a canonical pairing,
- $(TM \oplus T^*M \xrightarrow{\pi} M, pr_1, [\cdot,\cdot]_D^H, \langle\cdot,\cdot\rangle)$, for $[\cdot,\cdot]_D^H$ an $H-$twisted Dorfman bracket defined as $[X+\xi, Y+\eta]_D^H := [X+\xi, Y+\eta]_D - H(X,Y,\cdot)$ where $H \in \Omega^3(M)$

Courant Algebroid Connection

Courant algebroid connection is a map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ for which $\nabla(fe, e') = f\nabla(e, e')$ and $\nabla(e, fe') = f\nabla(e, e') + (\rho(e) \cdot f)e'$ for each $e, e' \in \Gamma(E)$ and ∇ is compatible with the Courant algebroid metric.

Example:
For Courant algebroid $(TM \oplus T^*M \xrightarrow{\pi} M, \rho, [\cdot,\cdot]_D, \langle\cdot,\cdot\rangle_E)$ we may define a Courant algebroid connection ∇ in the following way:

$$\langle \nabla_a b, c\rangle_E := \langle [a,b]_D - [a,b]_E, c\rangle_E$$

for $a, b, c \in \Gamma(E)$ and $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ an arbitrary skew-symmetric bracket with Leibniz rule.
(Here E is a shorthand notation for $TM \oplus T^*M$.)

Conclusion

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- This formalism also allows for the generalization of linear algebras.
- There is a theorem of Roytenberg and Ševera giving us a one-to-one correspondence between Courant algebroids and Poisson manifolds with a nilpotent vector field on them.

"My name's Jeff." - Jeff

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