One-jets of groupoids – bisection construction approach

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Outline

- Motivation
- General theory of higher bisection groups
- Presentation of automorphism groups
- Lie-Rinehart n-pairs
- Application to higher principal connections
- Further work

Motivation

- Write down an explicit formula for a 1-jet of a general Lie n-groupoid is a hard task (still not completely solved).
- Most of the obstacles vanish if one deals with a *strict* Lie n-groups instead.
- Surprisingly, there is a way how one can encapsulate *most of data* from Lie 1-groupoids to Lie 1-groups – by means of taking a bisection construction. [*A. Schmeding, Ch. Wockel; (Re)constructing Lie groupoids from their bisections and applications to prequantisation*]

  Conjecture below is a try for vast generalization in the category of higher Lie algebroids[1]:

Conjecture: Lie-Rinehart n-pairs associated to 1-jets of Lie n-groupoids are *homotopy equivalent* to Lie-Rinehart n-pairs associated to 1-jets of bisection groups of these Lie n-groupoids.

[1] The conjecture is a formalization of ideas discussed **here**.

Motivation

Our program is the following:

- to describe Lie-Rinehart n-pairs associated to 1-jets of bisection groups of strict Lie n-groupoids.

- apply the results on a theory of higher principal connections, compare them with known formulae, come up with potentially unknown ones...

Motivation

Our program is the following:

- to describe Lie-Rinehart n-pairs associated to 1-jets of bisection groups of strict Lie n-groupoids.
- apply the results on a theory of higher principal connections, compare them with known formulae, come up with potentially unknown ones...

- Fact 1: Any 1-groupoid object in a (higher) topos $\mathbf{H}$ can be seen as an effective epimorphism $f : \mathfrak{G}_0 \twoheadrightarrow \mathfrak{G}_{-1}$ in $\mathbf{H}$. [Lurie; Higher Topos Theory]

In a great generality, we have the following definition of bisection groups (taken from **nLab**).

Definition: A **bisection group** $\mathcal{B}(f)$ associated to an effective epimorphism $f : \mathfrak{G}_0 \twoheadrightarrow \mathfrak{G}_{-1}$ in an ∞-topos $\mathbf{H}$ is a group object in $\mathbf{H}$ defined as:

$$\mathcal{B}(f) := \prod_{\mathfrak{G}_{-1}} \mathrm{Aut}_{\mathbf{H}_{/\mathfrak{G}_{-1}}}(f) \ .$$

(where $\prod_{\mathfrak{G}_{-1}} : \mathbf{H}_{/\mathfrak{G}_{-1}} \to \mathbf{H}_{/*} \cong \mathbf{H}$ is a dependent product functor, i.e. the right adjoint to the base change functor)

General theory of higher bisection groups

- Fact 2: Due to the general formula for computing simplicial hom-space in (higher) slice topoi there is a more constructive characterization of $\mathcal{B}(f)$ saying that $\mathcal{B}(f)$ is an universal vertex of $(\infty, 1)$-pullback diagram

$$\begin{tikzcd} & \mathrm{Aut}_{\mathbf{H}}(\mathfrak{G}_0) \arrow[d, "\underline{\mathrm{hom}}_{\mathbf{H}}(\mathfrak{G}_0, f)"] \\ * \arrow[r, "\vdash f"] & \underline{\mathrm{hom}}_{\mathbf{H}}(\mathfrak{G}_0, \mathfrak{G}_{-1}) \end{tikzcd}$$

[D.Fiorenza, Ch. L. Rogers, U.Schreiber; Higher U(1)-gerbe connections in geometric prequantization]

More or less intrigue examination of the general construction given above produces more or less obvious examples of higher bisection groups. From now on $\mathbf{H} := \mathsf{Sh}_{(\infty,1)}(\mathsf{Cart})$.

Examples:

- If we consider $M \twoheadrightarrow *$ an effective epimorphism encoding a Pair groupoid $\mathsf{Pair}(M)$ for M a differentiable stack we get $\mathcal{B}(f) \cong \mathsf{Aut}(M)$, where $\mathsf{Aut}(M)$ is a $\mathbf{H}$-valued ∞-automorphism group on a differentiable stack.
- If $f : M \to \mathsf{coim}(g)$ an effective epimorphism (obtained from a classifying cocycle $g : M \to \mathbf{B}G$) encoding a higher Atiyah groupoid $\mathsf{At}(g)$ we get $\mathcal{B}(f) \cong \mathsf{Aut}(g)$, where $\mathsf{Aut}(g)$ is an $\mathbf{H}$-valued ∞-automorphism group on a (higher) principal bundle given by g.

Presentation of automorphism groups

- Abstractly speaking an ∞-automorphism group $\mathrm{Aut}(A)$ on some object A of a closed monoidal ∞-category $\mathcal{C}$ is a maximal subobject of $\underline{\mathrm{hom}}_{\mathcal{C}}(A, A)$ on all morphisms that are equivalences.
- If $\mathcal{C}$ is $\mathbf{H}$, then we have $\underline{\mathrm{hom}}_{\mathcal{C}}(A, A)[U] \cong \mathrm{hom}_{\mathcal{C}}(A \times \mathcal{Y}(U), A)$.
- In our case $\mathrm{hom}_{\mathcal{C}}(A, A) \cong \mathbb{R}\mathrm{hom}_{\mathsf{M}}(A, A) \cong \mathrm{hom}_{\mathsf{M}}(CA, FA) \in \mathrm{sSet}$, where M is a simplicial model category presenting $\mathcal{C}$.
- Even more concretely we have $\mathsf{M} = \mathrm{sPSh}_{\text{LOC.-PROJ.}}$ in case of presenting $\mathrm{Aut}(M)$ and $\mathrm{Aut}(g)$.

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Presentation of automorphism groups

From the previous observations we gain some corollaries:

Lemma: Let us have M a simplicial model category $\mathsf{sPSh}_{\mathrm{LOC.\text{-}PROJ.}}$ presenting $\mathbf{H}$. In this model category $\mathsf{Aut}_{\mathbf{H}}(M)$ (for M presented as an n-truncated Kan simplicial manifold a.k.a. Lie n-groupoid) form the following simplicial diffeological group:

- $\mathsf{Aut}(M)_q := \{\alpha : M \otimes \Delta^{[q]} \to M | \alpha \circ (\mathsf{id}_M \otimes \partial_I^{\times q}) \in \mathsf{Diff}(M)\}$
- $\mathsf{f}_i := (-) \circ (\mathsf{id}_M \otimes \delta_i)$, $\mathsf{d}_i := (-) \circ (\mathsf{id}_M \otimes \sigma_i)$
- $(\alpha \star \tilde{\alpha})(m \otimes u) := \alpha(\tilde{\alpha}(m \otimes u) \otimes u)$

Lemma: Let us have M a simplicial model category $\mathsf{sPSh}_{\text{LOC.-PROJ.}}$ presenting $\mathbf{H}$ and g being presented as a higher strict principal bundle $\pi : P \to M$ with a principal action $\mu : \mathsf{G} \times \mathsf{P} \to \mathsf{P}$ (for M and P presented as n-truncated Kan simplicial manifolds). In this model category $\mathsf{Aut}_{\mathbf{H}}(g)$ is presented as follows:

- $\mathsf{Aut}(g)_q := \{\beta_1 : P \otimes \Delta^{[q]} \to P; \beta_0 : M \otimes \Delta^{[q]} \to M | \beta_1 \circ (\mathsf{id}_P \otimes \partial_I^{\times q}) \in \mathsf{Diff}(P); \beta_0 \circ (\mathsf{id}_M \otimes \partial_I^{\times q}) \in \mathsf{Diff}(M); \pi \circ \beta_1 = \beta_0 \circ (\pi \otimes \mathsf{id}_{\Delta^q}); \beta_1 \circ \tilde{\mu} = \mu \circ (e_{\mathsf{G}} \times \beta_1)\}$
- $\mathsf{f}_i := (-) \circ (\mathsf{id}_M \otimes \delta_i)$, $\mathsf{d}_i := (-) \circ (\mathsf{id}_M \otimes \sigma_i)$
- $(\beta_1, \beta_0) \star (\tilde{\beta}_1, \tilde{\beta}_0)((p \otimes u); (m \otimes u)) = (\beta_1(\tilde{\beta}_1(p \otimes u) \otimes u); \beta_0(\tilde{\beta}_0(m \otimes u) \otimes u))$

Now, let us specialize onto certain subcategory (homotopy equivalent to!) of *higher strict principal bundles*. Most of the twisted Cartesian product theory of bundles is taken from [P.May; Simplicial Objects in Algebraic Topology].

Definition: Let $\mathsf{M}_\bullet$ and $\mathsf{Y}_\bullet$ be Kan simplicial manifolds and let $\mathsf{G}_\bullet$ be a simplicial Lie group. Furthermore, let $\triangleleft : \mathsf{Y}_\bullet \times \mathsf{G}_\bullet \to \mathsf{Y}_\bullet$ be a right-action of $\mathsf{G}_\bullet$ on $\mathsf{Y}_\bullet$ and let $\tau : \mathsf{M}_\bullet \to \mathsf{G}_{\bullet-1}$ be a *twisting function*. Then, the **twisted Cartesian product**, denoted by $\mathsf{Y}_\bullet \times_\tau \mathsf{M}_\bullet$, is a Kan simplicial manifold

$$(\mathsf{Y}_\bullet \times_\tau \mathsf{M}_\bullet)_n \ := \ \mathsf{Y}_n \times \mathsf{M}_n$$

with face and degeneracy maps being defined by

$$\mathsf{f}_i^n(y,m) \ := \begin{cases} (\mathsf{f}_0^n(y) \triangleleft \tau(m), \mathsf{f}_0^n(m)) & \text{for } i = 0 \\ (\mathsf{f}_i^n(y), \mathsf{f}_i^n(m)) & \text{else} \end{cases},$$

$$\mathsf{d}_i^n(y,m) \ := \ (\mathsf{d}_i^n(y), \mathsf{d}_i^n(m)).$$

- If the group action is free and transitive, then we have (non-canonical) identification $\mathsf{G}_\bullet \cong \mathsf{Y}_\bullet$ and we call this space *principal twisted Cartesian product*.

- A fibration $\pi : (\mathsf{G}_\bullet \times_\tau \mathsf{M}_\bullet) \to \mathsf{M}_\bullet$ defined as

$$\pi(g, m) := m$$

together with a *left* $\mathsf{G}_\bullet$-action $\triangleright : \mathsf{G}_\bullet \times (\mathsf{G}_\bullet \times_\tau \mathsf{M}_\bullet) \to (\mathsf{Y}_\bullet \times_\tau \mathsf{M}_\bullet)$ defined as

$$h \triangleright (g, m) := (h \cdot g, m)$$

is called **principal twisted Cartesian product bundle**.

Presentation of automorphism groups

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Presentation of automorphism groups

Observation: If our principal bundle is a principal twisted Cartesian product bundle, then a simplicial group $\mathsf{Aut}(g)$ is in simplicial degree q described as a space of pairs $\{(\Psi^{[q]}_\bullet, \Phi^{[q]}_\bullet)\}$ (where nth component of each element is a pair of smooth maps $\Psi^{[q]}_n : (M \otimes \Delta^{[q]})_n \to \mathsf{G}_n$, $\Phi^{[q]}_n : (M \otimes \Delta^{[q]})_n \to M_n$) satisfying the equations:

$$
\begin{aligned}
\Psi^{[q]}_n(\mathsf{f}_0 m, \mathsf{f}_0 u) \cdot \tau(m) &= \tau(\Phi^{[q]}_{n+1}(m,u)) \cdot \mathsf{f}_0 \Psi^{[q]}_{n+1}(m,u) \ , \\
\mathsf{f}_{i>0} \Psi^{[q]}_{n+1}(m,u) &= \Psi^{[q]}_n \mathsf{f}_{i>0}(m,u) \ , \\
\mathsf{d}_j \Psi^{[q]}_n(m,u) &= \Psi^{[q]}_{n+1} \mathsf{d}_j(m,u) \ , \\
\mathsf{f}_i \Phi^{[q]}_{n+1}(m,u) &= \Phi^{[q]}_n \mathsf{f}_i(m,u) \ , \\
\mathsf{d}_j \Phi^{[q]}_n(m,u) &= \Phi^{[q]}_{n+1} \mathsf{d}_j(m,u) \ .
\end{aligned}
$$

Presentation of automorphism groups

... together with face and degeneracy maps on pairs $\{(\Psi_\bullet^{[q]}, \Phi_\bullet^{[q]})\}$:

$$\mathsf{f}_i^{[q]} := ((-)\circ(\mathrm{id}_M \otimes \partial_i^q), (-)\circ(\mathrm{id}_M \otimes \partial_i^q))$$

$$\mathsf{d}_i^{[q]} := ((-)\circ(\mathrm{id}_M \otimes \delta_i^q), (-)\circ(\mathrm{id}_M \otimes \delta_i^q))$$

and a group composition law:

$$(\Psi_\bullet^{[q]}, \Phi_\bullet^{[q]}) * (\tilde{\Psi}_\bullet^{[q]}, \tilde{\Phi}_\bullet^{[q]}) := (\Psi_\bullet^{[q]} \underset{\tilde{\Phi}_\bullet^{[q]}}{\star} \tilde{\Psi}_\bullet^{[q]}, \Phi_\bullet^{[q]} \star \tilde{\Phi}_\bullet^{[q]}),$$

where

$$(\Psi_\bullet^{[q]} \underset{\tilde{\Phi}_\bullet^{[q]}}{\star} \tilde{\Psi}_\bullet^{[q]})_n(m,u) := \Psi_n^{[q]}(\tilde{\Phi}_n^{[q]}(m,u),u)\cdot\tilde{\Psi}_n^{[q]}(m,u),$$

$$(\Phi_\bullet^{[q]} \star \tilde{\Phi}_\bullet^{[q]})_n(m,u) := \Phi_n^{[q]}(\tilde{\Phi}_n^{[q]}(m,u),u).$$

Lie-Rinehart n-pairs

Definition: A Lie-Rinehart n-pair is a triple $(A, L, \mathbf{X})$, where A is a commutative algebra, L is a graded A-module, and $\mathbf{X}$ is a degree 1 *multiderivation* which satisfies $\mathbf{X}_0 = 0$ *(non-curved case)* and $\mathbf{X} \circ \mathbf{X} = 0$ *(Leibniz rule and higher Lie bracket commutator rule generalized)*.

Details are rather technical, for more see

- [D.Pištalo, Pro-nilpotently Extended Dgca-s and SH Lie-Rinehart Pairs],
- [L. Vitagliano, Representations of Homotopy Lie-Rinehart Algebras]

Lie-Rinehart n-pairs

As we have seen, for a reconstruction of any Lie-Rinehart n-pair we need to recast:

- a commutative algebra A (what is always $\mathrm{C}^\infty(M)$ for (any cofibrant resolutions) of an ordinary smooth manifold)
- a multiderivation $\mathbf{X}$ (will not be discussed today)
- A-module which underlines L_∞ algebra structure of L. This is what will be discussed now...

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Lie-Rinehart n-pairs

Since we already have (at least implicit) description of simplicial automorphism groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(g)$ we can ask for their dg Lie algebras. The following statement addresses this task.

Theorem.(Jurčo) For any strict simplicial Lie/diffeological n-group $\mathsf{G}_\bullet$ its 1-jet corresponds to a dg Lie algebra obtained from a hyper-crossed complex construction applied on simplicial degree-wise Lie differentiation of $\mathsf{G}_\bullet$.

To be found in [B. Jurčo, From Simplicial Lie Algebras and Hypercrossed Complexes to Differential Graded Lie Algebras via 1-jets].

So to obtain (again, implicit datum for) a dg Lie algebra in question it suffices to linearize our equations (take a tangent space above bisection group identity) and apply hyper-crossed complex construction on them.

Let $G_\bullet$ be a simplicial Lie/diffeological group and $\mathfrak{g}_\bullet$ be its simplicial Lie algebra obtained from simplicial degree-wise Lie differentiation. The **hyper-crossed complex** $(N\mathfrak{g}_\bullet, \partial_\bullet, r_\bullet, f_{\bullet,\bullet})$ of $\mathfrak{g}_\bullet$ is the chain complex

$$
\begin{aligned}
N\mathfrak{g}_0 &:= \mathfrak{g}_0 \ , \\
N\mathfrak{g}_n &:= \bigcap_{i=1}^{n} \ker \mathsf{f}_i^{\mathfrak{g}_n} \quad n \geq 1 \ , \\
\partial_n &:= \mathsf{f}_0^{\mathfrak{g}_n} \restriction_{N\mathfrak{g}_n} \quad n \geq 1 \ ,
\end{aligned}
$$

together with an action $r_\bullet : N\mathfrak{g}_n \times N\mathfrak{g}_0 \to N\mathfrak{g}_n$ for $n > 0$ and a Peiffer pairing $f_{\bullet,\bullet}$ defined for instance in [I. Akca, Z. Arvasi, Simplicial and Crossed Lie Algebras]. From this complex we can reconstruct differential, grading and Lie brackets of the resulting differential graded Lie algebra. But not now!

Lie-Rinehart n-pairs

A $C^\infty(M)$-module underlying to a Lie-Rinehart n-pair of an Atiyah-Lie n-algebroid (above a twisted cartesian product bundle encoded in a twisting τ for some strict $G_\bullet$) is a graded abelian group of pairs $\{(\dot\Psi^{[q]}_\bullet, \dot\Phi^{[q]}_\bullet)\}$ (where nth component of $(q+1)$-degree element is a pair of smooth maps $\dot\Psi^{[q]}_n : (M \otimes \Delta^{[q]})_n \to \mathrm{Lie}(G_n)$, $\dot\Phi^{[q]}_n : (M \otimes \Delta^{[q]})_n \xrightarrow{\sigma} TM_n$) satisfying the following equations:

$$\mathrm{Ad}_{\tau(m)}(\dot\Psi^{[q]}_n(\mathrm{f}_0 m, \mathrm{f}_0 u) + T\tau_n \circ \dot\Phi^{[q]}_n(m,u)) = (\mathrm{T}_{e_{G_n}}\mathrm{f}_0)(\dot\Psi^{[q]}_{n+1}(m,u)) \ ,$$

$$(\mathrm{T}_{e_{G_{n+1}}}\mathrm{f}_{i>0})(\dot\Psi^{[q]}_{n+1}(m,u)) = \dot\Psi^{[q]}_n \mathrm{f}_{i>0}(m,u) \ ,$$

$$(\mathrm{T}_{e_{G_n}}\mathrm{d}_j)(\dot\Psi^{[q]}_n(m,u)) = \dot\Psi^{[q]}_{n+1}\mathrm{d}_j(m,u) \ ,$$

$$T\mathrm{f}_i \dot\Phi^{[q]}_{n+1}(m,u) = \dot\Phi^{[q]}_n \mathrm{f}_i(m,u) \ ,$$

$$T\mathrm{d}_j \dot\Phi^{[q]}_n(m,u) = \dot\Phi^{[q]}_{n+1}\mathrm{d}_j(m,u) \ ,$$

$$\dot\Psi^{[q]}_n(m, \delta_i u) = 0 \ ,$$

$$\dot\Phi^{[q]}_n(m, \delta_i u) = 0 \ .$$

...together with a $C^\infty(M)$ action $\propto$ defined on components:

$$(f \propto \dot{\Psi}_n^{[q]})(m,u) = f(m) \cdot \dot{\Psi}_n^{[q]}(m,u),$$
$$(f \propto \dot{\Phi}_n^{[q]})(m,u) = f(m) \cdot \dot{\Phi}_n^{[q]}(m,u),$$

where $f \in C^\infty(M)$.

Application to higher principal connections

- A nice category for accommodating a reasonable definition of a higher flat parallel transporter is $\mathrm{Ho}(\mathrm{Grpd}(\mathbf{H}))$, the homotopy category of groupoid objects in $\mathbf{H}$.

- Higher connections are then algebraic data obtained from a higher parallel transporter under application of the (homotopy) Lie functor.

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Application to higher principal connections

As a warm up let us state one useful theorem about higher flat parallel transporters defined in $\mathsf{Ho}(\mathsf{Grpd}(\mathbf{H}))$:

Theorem.(N.) (only one direction of a full equivalence)
Assume we are given some $g \in \pi_0\mathrm{hom}_{\mathbf{H}}(M, \mathbf{B}G)$. Then we claim that if $\nabla \in \pi_0\mathrm{hom}_{\mathbf{H}}(\Pi(\mathrm{M}), \mathbf{B}G)$ is over g then it uniquely determines a morphism $\sigma \in \mathsf{Ho}(\mathsf{Grpd}(\mathbf{H}))$ such that σ is a lift of the diagram below with the property $\sigma_0 = id_M$.

$$\begin{tikzcd} \mathbf{\Pi}(\mathrm{M}) \arrow[rr] \arrow[dr, dashed, "\sigma"'] & & \mathbf{Pair}(\mathrm{M}) \\ & \mathbf{At}(\mathrm{g}) \arrow[ur] & \end{tikzcd}$$

Application to higher principal connections

When the Lie functor (a.k.a. 1-jet functor) applied on the lifting problem above (on a 2-cell in Grpd$(\mathbf{H})$ witnessing the commutativity in the homotopy category) we get a lifting problem in category of Lie-Rinehart n-pairs. But, according to our conjecture 1-jets of these groupoids (in terms of Lie-Rinehart n-pairs) are homotopy equivalent to Lie-Rinehart n-pairs obtained from their bisection groups! Thus we want to examine the lifting problem

$$\begin{tikzcd} \mathsf{aut}(M) \arrow[rr] \arrow[dr, dashed, "\sigma"'] & \arrow[d, phantom, "\Downarrow"] & \mathsf{aut}(M) \\ & \mathsf{aut}(g) \arrow[ur] & \end{tikzcd}$$

Application to higher principal connections

And because Lie-Rinehart n-pairs form a subcategory of differential graded commutative algebras under the Chevalley-Eilenberg complex construction we can modify our lifting problem even further

$$
\begin{tikzcd}
\mathsf{CE}(\mathsf{aut}(M)) & & \mathsf{CE}(\mathsf{aut}(M)) \arrow[ll] \arrow[dl] \\
& \mathsf{CE}(\mathsf{aut}(g)) \arrow[ul, dashed, "\mathsf{CE}(\sigma)"] \arrow[u, phantom, "\Downarrow"] &
\end{tikzcd}
$$

Application to higher principal connections

Here we have some observations:

- Moduli space of all such solutions CE(σ) must be a subspace of CE(aut(M)) $\otimes$ $(\Gamma($CE(aut$(g))))^*$ (Γ is a symbol for restriction on algebraic generators), in most traditional cases $\Omega^\bullet(M) \otimes$ Lie(G)
- Compatibility of CE(σ) with differentials implies Maurer-Cartan equation, what is traditionally interpreted as a flatness condition
- Defining equations (their CE counterpart respectively) for Lie-Rinehart n-pairs aut(M) and aut(g) take the role of gluing conditions/local gauge transformations.

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Further work

There is still much to work on, namely:

- Work out examples for higher connections with strict n-group as a structure group
- Examine cases when M is a higher base space. It leads to differential associative algebra resolution and switch from Lie-Rinehart n-pairs to SH Lie-Rinehart pairs organizing derived Lie algebroids.
- Fully formulate the bisection group construction for other/general higher Lie groupoids as to get closer to proving the original conjecture

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THANK YOU!