Quantum exterior algebras and torsion free bimodule connections

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(Joint work with Alessandro Carotenuto and Junaid Razzaq)

1 Introduction

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- The study of non-commutative algebras with algebraic properties similar to those of $C^\infty(M)$, for M a smooth manifold!

- So how can we produce an analogue of vector fields and one-forms for a noncommutative algebra?

2 Differential Calculi

Definition

A *first-order differential calculus* over an algebra B is a B-bimodule $\Omega^1(B)$ with a linear map

$$\mathrm{d} : B \to \Omega^1(B)$$

such that $\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}b$, for all $a, b \in B$, and the multiplication map $B \otimes \mathrm{d}B \to \Omega^1(B)$ is surjective.

- What about higher forms? That is, can we extend to get a differential graded algebra:

$$B \to \Omega^1(B) \to \Omega^2(B) \to \cdots$$

such that $\mathrm{d}^2 = 0$ and d satisfies a graded Leibniz rule.

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- In general, taking an exterior algebra is not a good idea.

- Take the tensor algebra

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- Can we find a B-sub-bimodule $N^{(2)} \subseteq \Omega^1(B) \otimes \Omega^1(B)$, such that the quotient

$$\mathcal{T}(\Omega^1(B))/\langle N^{(2)} \rangle$$

has the (necessarily unique) structure of a differential graded algebra d extending $\mathrm{d} : B \to \Omega^1(B)$?

- The answer is yes! Moreover, there is ”maximal” way to do this!

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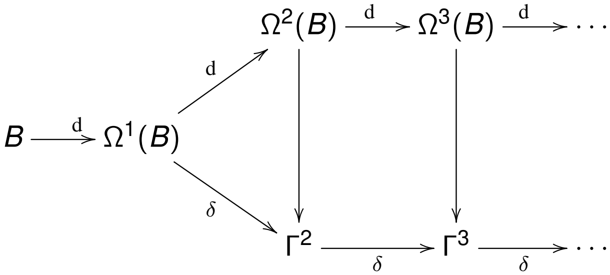
- There exists a unique $N^{(2)}$ such that for any other extension to a dga $\Gamma^\bullet$, we have the commutative diagram



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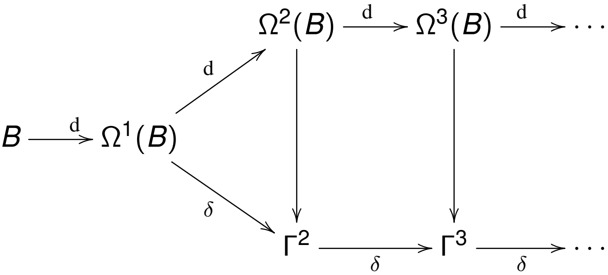

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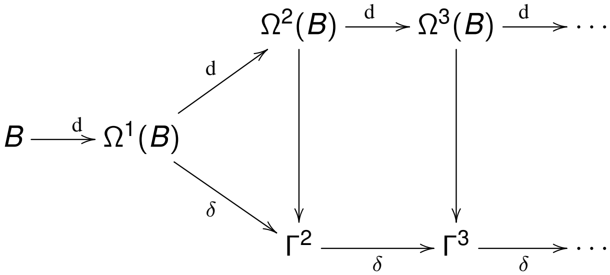

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Example

For the space of one forms $\Omega^1(B)$, its maximal prolongation is the usual exterior algebra construction of the de Rham complex.

3: Torsion-Free Bimodule Connections

Definition

For a left B-module $\mathcal{F}$, a *connection* is a linear map $\nabla : \mathcal{F} \to \Omega^1(B) \otimes_B \mathcal{F}$ such that

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla(f), \qquad \text{for all } b \in B,\ f \in \mathcal{F}.$$

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Definition

A connection ∇ for $\mathcal{F}$ is said to be a *bimodule connection* if

$$\nabla(fb) = \nabla(f)b + \sigma(f \otimes \mathrm{d}b), \qquad \text{for all } b \in B,\ f \in \mathcal{F},$$

for a (necessarily unique) bimodule map

$$\sigma : \mathcal{F} \otimes \Omega^1(B) \to \Omega^1(B) \otimes_B \mathcal{F}.$$

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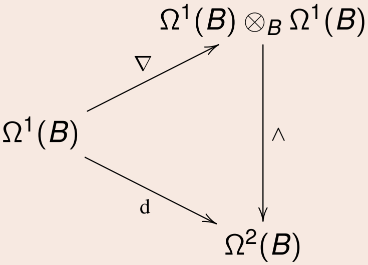
that is, such that the following diagram commutes



Theorem

(A. Carotenuto, RÓB, J. Razzaq) For any torsion-free bimodule connection $\nabla : \Omega^1(B) \to \Omega^1(B) \otimes_B \Omega^1(B)$*, with bimodule map* σ*, it holds that* $N^{(2)}$ *is generated as a* B*-bimodule by* $G_1 \cup G_2$*, where*

$$G_1 := \left\{ \omega \otimes \nu + \sigma(\omega \otimes \nu) \,|\, \omega,\, \nu \in \Omega^1(B) \right\}$$

$$G_2 := \left\{ \nabla(\mathrm{d}b) \,|\, b \in B \right\}$$

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Definition

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Proposition

- *Let* $\Omega^1(B)$ *be a first-order differential calculus over* B *endowed with a "compatible" left* A*-coaction. Denote* $B^+ = \ker(\varepsilon_A) \cap B$*, and assume that* $\Omega^1(B)B^+ = B^+\Omega^1(B)$*.*

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- *Then for any equivariant bimodule connection, with associated bimodule map* σ*, it holds that*

$$\sigma(\mathrm{d}b \otimes \mathrm{d}c) = \mathrm{d}(b_{(3)}cS(b_{(2)})) \otimes \mathrm{d}b_{(1)}, \quad \textit{for all } b, c \in B.$$

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- Note: σ is independent of the choice of connection.

5: Drinfeld–Jimbo Quantum Flag Manifolds

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- For any $q \in \mathbb{R}_{>0}$, the Drinfeld–Jimbo quantum groups are a dual pair

$$\mathcal{O}_q(G) \times U_q(\mathfrak{g}) \to \mathbb{C},$$

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of (co)quasitriangular Hopf algebras.
- When $q = 1$, we recover the algebra of representable (polynomial) functions $\mathcal{O}(G)$, and the universal enveloping algebra $U(\mathfrak{g})$.

- For S a subset of simple roots, we have the *quantum Levi subalgebra*

$$U_q(\mathfrak{l}_S) := \left\langle K_i, E_j, F_j \,|\, i = 1, \ldots, r; j \in S \right\rangle$$

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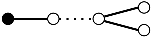
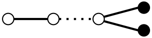
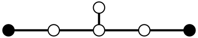
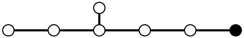
Definition

For S a subset of simple roots of $\mathfrak{g}$, the corresponding *quantum flag manifold* is the invariant subspace

$$\begin{aligned} \mathcal{O}_q(G/L_S) :=& \mathcal{O}_q(G)^{U_q(\mathfrak{l}_S)} \\ =& \left\{ g \in \mathcal{O}_q(G) | g \triangleleft X = \varepsilon(X) g, \forall X \in U_q(\mathfrak{l}_S) \right\}. \end{aligned}$$

Compact Quantum Hermitian Symmetric Spaces

A_n		$\mathcal{O}_q(\mathrm{Gr}_{n,r})$	quantum Grassmanian
B_n		$\mathcal{O}_q(\mathrm{Q}_{2n+1})$	odd quantum quadric
C_n		$\mathcal{O}_q(\mathrm{L}_n)$	symmetric q.-Lagrangian Grassmannian
D_n		$\mathcal{O}_q(\mathrm{Q}_{2n})$	even quantum quadric
D_n		$\mathcal{O}_q(\mathrm{S}_n)$	quantum spinor variety
E_6		$\mathcal{O}_q(\mathbb{OP}^2)$	quantum Cayley plane
E_7		$\mathcal{O}_q(\mathrm{F})$	quantum Freudenthal variety

Theorem (Heckenberger, Kolb '06)

For each compact quantum Hermitian symmetric flag manifold $\mathcal{O}_q(G/L_S)$, there exist precisely two irreducible $U_q(\mathfrak{g})$-covariant first-order differential calculi for $\Omega_q^\bullet(G/L)$:

$$\Omega_q^1(G/L_S) := \Omega_q^{(1,0)} \oplus \Omega_q^{(0,1)}.$$

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$$\Omega_q^1(G/L_S) := \Omega_q^{(1,0)} \oplus \Omega_q^{(0,1)}.$$

- In the $q = 1$ limit these correspond to the decomposition of complexified 1-forms into its holomorphic and anti-holomorphic summands"

$$\Omega^1(G/L_S) \simeq \Omega^{(1,0)} \oplus \Omega^{(0,1)}.$$

- How to describe the maximal prolongation of $\Omega_q^1(G/L_S)$?

To do this we need some notation: With respect to the index set $J := \{1, \dots, \dim(V_{\varpi_s})\}$:

$$\widehat{R}_{V_{\varpi_s}, V_{\varpi_s}}(v_i \otimes v_j) =: \sum_{k,l \in J} \widehat{R}_{ij}^{kl} \, v_k \otimes v_l,$$

$$\widehat{R}_{V_{-w_0(\varpi_s)}, V_{\varpi_s}}(f_i \otimes v_j) =: \sum_{k,l \in J} \acute{R}^{-kl}_{ij} \, v_k \otimes f_l,$$

$$\widehat{R}_{V_{\varpi_s}, V_{-w_0(\varpi_s)}}(v_i \otimes f_j) =: \sum_{k,l \in J} \grave{R}^{-kl}_{ij} \, f_k \otimes v_l,$$

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Moreover, we denote by $\widehat{R}^-$, $\acute{R}$, $\grave{R}$, and $\check{R}^-$, the inverse matrices of $\widehat{R}$, $\acute{R}^-$, $\grave{R}^-$, and $\check{R}$ respectively.

The subbimodule $N^{(2)}$ can now be given in terms of the standard matrix generator z $:= (z_{ij})_{(ij)}$:

First are the *holomorphic relations*

$$\widehat{Q}_{12}\acute{R}_{23}\partial\mathbf{z}\wedge\partial\mathbf{z}=0, \qquad \check{P}_{34}\acute{R}_{23}\partial\mathbf{z}\wedge\partial\mathbf{z}=0, \qquad (1)$$

where we have used leg notation, and have denoted

$$\widehat{Q}:=\widehat{R}+q^{(\varpi_s,\varpi_s)-(\alpha_x,\alpha_x)}\mathrm{id}, \qquad \check{P}:=\check{R}-q^{(\varpi_s,\varpi_s)}\mathrm{id}.$$

Second are the *anti-holomorphic relations*

$$\widehat{P}_{12}\acute{R}_{23}\overline{\partial}\mathbf{z}\wedge\overline{\partial}\mathbf{z}=0, \qquad \check{Q}_{34}\acute{R}_{23}\overline{\partial}\mathbf{z}\wedge\overline{\partial}\mathbf{z}=0, \qquad (2)$$

where we have again used leg notation, and have denoted

$$\widehat{P}:=\widehat{R}-q^{(\varpi_s,\varpi_s)}\mathrm{id}, \qquad \check{Q}:=\check{R}+q^{(\varpi_s,\varpi_s)-(\alpha_x,\alpha_x)}\mathrm{id}.$$

Finally, we have the *cross-relations*

$$\overline{\partial}\mathbf{z} \wedge \partial\mathbf{z} = -q^{-(\alpha_x,\alpha_x)} T^{-}_{1234} \partial\mathbf{z} \wedge \overline{\partial}\mathbf{z} + q^{(\varpi_s,\varpi_s)-(\alpha_x,\alpha_x)} z C_{12} T^{-}_{1234} \partial\mathbf{z} \wedge \overline{\partial}\mathbf{z},$$

where we have again used leg notation, and have denoted

$$T^{-}_{1234} := \grave{R}^{-}_{23} \widehat{R}^{-}_{12} \check{R}_{34} \acute{R}_{23}, \qquad C_{kl} := \sum_{i=1}^{\dim(V_{\varpi_s})} \grave{R}^{-ii}_{kl}.$$

Theorem (FDG-AK-RÓB-PS-KRS '21)

Each Heckenberger–Kolb calculus $\Omega^1_q(G/L_S)$ admits a unique $U_q(\mathfrak{g})$-equivariant connection

$$\nabla : \Omega^1_q(G/L_S) \to \Omega^1_q(G/L_S) \otimes_B \Omega^1_q(G/L_S).$$

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Theorem (AK-JB-RÓB-BG '24)

Each ∇ is a bimodule connection.

Theorem (AC-JR-RÓB)

For each Heckenberger–Kolb calculus $\Omega^1_q(G/L_S)$, it holds that

$$N^{(2)} = \left\{ \omega \otimes \nu + \sigma(\omega \otimes \nu) \,\middle|\, \omega, \nu \in \Omega^1_q(G/L_S) \right\}.$$

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Corollary

Indeed, $N^{(2)}$ is spanned as an $\mathcal{O}_q(G/L_S)$-bimodule by the elements

$$\mathrm{d}b \otimes \mathrm{d}c + \mathrm{d}(b_{(3)} c S(b_{(1)}) \otimes \mathrm{d}b \qquad \textit{for} \qquad b, c \in \mathcal{O}_q(G/L_S).$$