$SL(2)$ theory and quantum computing

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EUROPSKI STRUKTURNI
I INVESTICIJSKI FONDOVI

Operativni program
KONKURENTNOST
I KOHEZIJA

Europska unija
Zajedno do fondova EU

Joint work in progress with:

Jing-Song Huang, CUHK Shenzhen

Soo Teck Lee, National University of Singapore

Previous work

Our work is for now mostly concerning the case of 3 qubits, and for this case all of our results are covered by the existing literature. Some of the authors are

Brylinski
Meyer-Wallach
Piatetski-Shapiro and Rallis
Brion
Baldoni-Vergne
Briand-Luque-Thibon
Walter
Walter-Doran-Gross-Christandl
Le Paige (1881!)

Previous work

We however believe our method is going to prove slightly better and we will be able to go further, increasing the number of qubits, and/or going to larger dimensions of factors (qudits).

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Another remark is that with our approach we can also understand the skew version, where the algebra of polynomials is replaced by the corresponding exterior algebra.

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One would like to classify k-qubit states modulo the action of $U(2)^{\times k}$.

As a preliminary step, one can consider orbits of $GL_2(\mathbb{C})^{\times k}$ on $(\mathbb{C}^2)^{\otimes k}$. These are algebraic varieties, given as zero sets of some polynomials in $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes k})$.

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From now on we assume $k = 3$.

Entanglement

$x \in (\mathbb{C}^2)^{\otimes 3}$ is decomposable (rank 1, pure, non-entangled) if it is possible to write it as

$$x = u \otimes v \otimes w$$

for some $u, v, w \in \mathbb{C}^2$.

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Clearly, the rank of any tensor is ≤ 8. In fact, it is ≤ 3 (and generically it is 2).

Note that rank is constant along $GL_2(\mathbb{C})^{\times 3}$ orbits.

Howe's $GL_n \times GL_k$ duality

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Let x_{ij} denote the coordinate function corresponding to $e_i \otimes e_j$.

Then the algebra of highest weight vectors in $\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^k)$ is a polynomial algebra freely generated by

$$x_{11}, \quad \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \quad \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}, \quad \cdots$$

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It is also easy to see they are algebraically independent (leading terms are x_{11}, $x_{11}x_{22}$, $x_{11}x_{22}x_{33}$, ... Here "leading" is meant wrt reverse lexicographic order on the indices.)

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It is also easy to see they are algebraically independent (leading terms are x_{11}, $x_{11}x_{22}$, $x_{11}x_{22}x_{33}$, ... Here "leading" is meant wrt reverse lexicographic order on the indices.)

It is harder to see that these determinants indeed generate the whole algebra of highest weight vectors. For this, Howe uses some nontrivial algebraic geometry argument.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

The most obvious highest weight vector is x_{111}; it has weight $(1,0) \otimes (1,0) \otimes (1,0)$ and thus generates a $2^3 = 8$-dimensional representation. This representation is of course $(\mathbb{C}^2)^{\otimes 3}$, and we have accounted for all degree 1 polys.

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In degree 2 we have the highest weight vector x_{111}^2 of weight $(2,0) \otimes (2,0) \otimes (2,0)$, generating a 27-dimensional representation. Since

$$\dim \mathcal{P}^2 = \binom{2+7}{7} = 36,$$

we are missing 9 dimensions.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

The following polynomials are analogues of Howe's determinants:

$$d_{12} = \begin{vmatrix} x_{111} & x_{121} \\ x_{211} & x_{221} \end{vmatrix}, \quad d_{13} = \begin{vmatrix} x_{111} & x_{112} \\ x_{211} & x_{212} \end{vmatrix}, \quad d_{23} = \begin{vmatrix} x_{111} & x_{112} \\ x_{121} & x_{122} \end{vmatrix}.$$

(Keep one index 1, do Howe's determinant on the other 2 indices.)

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These are highest weight vectors of weights

$$(1,1)\otimes(1,1)\otimes(2,0), \quad (1,1)\otimes(2,0)\otimes(1,1), \quad (2,0)\otimes(1,1)\otimes(1,1),$$

bringing dimensions 3,3,3, exactly what we need.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

The leading terms of the above representations are the diagonals: $x_{111}x_{221}$, $x_{111}x_{212}$, $x_{111}x_{122}$. The (highest) weights are easily read off from these leading terms.

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The leading terms are again taken with respect to reverse lexicographical order of the indices. They behave nicely with respect to multiplication and are very important for everything that follows. The keyword here is SAGBI.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

Now we move to degree $N = 3$; so far we have

$$x_{111}^3, \quad x_{111}d_{12}, \quad x_{111}d_{13}, \quad x_{111}d_{23}.$$

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The total dimension these bring is 112, but $\dim \mathcal{P}^3 = \binom{3+7}{7} = 120$, so we are missing 8.

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Let $\omega_{12} = e_{21}^{(3)} d_{12} = x_{111}x_{222} + x_{112}x_{221} - x_{212}x_{121} - x_{211}x_{122}$.

(a sort of cubic determinant.)

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Furthermore, let $\eta_{12} = \frac{1}{2} e_{21}^{(3)} \omega_{12} = \begin{vmatrix} x_{112} & x_{122} \\ x_{212} & x_{222} \end{vmatrix}$.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

Then the polynomial

$$f_3 = x_{111}\omega_{12} - 2x_{112}d_{12}$$

is a highest weight vector (the corresponding representation is a PRV component in $\mathcal{P}^1 \otimes \mathcal{P}^2$.)

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So the representation has dimension 8, and we get what we wanted.

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In degree 4, we are missing a 1-dimensional representation, and we obtain it by setting

$$f_4 = \omega_{12}^2 - 4d_{12}\eta_{12}$$

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This leads to the relation

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We will see that this relation is closely related to multiplicity in the decomposition of $\mathcal{P}$.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

By now we have constructed highest weight vectors

$$x_{111}^a d_{12}^b d_{13}^c d_{23}^d f_3^e f_4^f ,$$

where $a, b, c, d, f \in \mathbb{Z}_+$ and $e = 0, 1$.

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where $a, b, c, d, f \in \mathbb{Z}_+$ and $e = 0, 1$.

The leading term of the above highest weight vector is

$$x_{111}^{a+b+c+d+2e+2f} x_{221}^b x_{212}^c x_{122}^d x_{222}^{e+2f}.$$

Note that a, b, c, d, e, f can be reconstructed. It follows that the above vectors are linearly independent.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

The corresponding highest weight is

$$(a+b+c+2d+2e+2f,\, b+c+e+2f) \otimes$$
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$$(a+2b+c+d+2e+2f, c+d+e+2f)$$

The dimension is

$$(a+2d+e+1)(a+2c+e+1)(a+2b+e+1).$$

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

To see that in this way we have obtained all highest weight vectors, we have to prove that for each degree N,

$$\sum_{\substack{a,\dots,f \text{ as above} \\ a+2b+2c+2d+3e+4f=N}} (a+2d+e+1)(a+2c+e+1)(a+2b+e+1) = \binom{N+7}{7}$$

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We have several proofs of this. The easiest one starts with the remark that both sides are polynomials in N of degree 7, so it is enough to check equality for $N = 0, 1, \dots, 7$.

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We already know it for $N \leq 4$, so we need to check $N = 5, 6, 7$. It is a bit tedious to do it by hand, but it is very easy for a computer.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

The representation of $GL_2^{\times 3}$ with highest weight

$$\lambda \otimes \mu \otimes \nu = (\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2) \otimes (\nu_1, \nu_2)$$

appears in $\mathcal{P}$ if and only if:

1. $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 = \nu_1 + \nu_2$ (=the degree), and
2. $m \leq M$, where

$$m = \max(0, \mu_2 - \lambda_2, \nu_2 - \lambda_2) \quad \text{and}$$
$$M = \min\left(\frac{\mu_2 + \nu_2 - \lambda_2 - e}{2}, \frac{\lambda_1 - \lambda_2 - e}{2}\right)$$

with $e \in \{0, 1\}$ being the parity of $\mu_2 + \nu_2 - \lambda_2$.

If these conditions are satisfied, then the multiplicity of $\lambda \otimes \mu \otimes \nu$ in $\mathcal{P}$ is equal to $[M] - m + 1$.

3 qubits: $GL_2^{\times 3}$-structure of $\mathcal{P} = \mathcal{P}((\mathbb{C}^2)^{\otimes 3})$

One sees that the multiplicity comes from interchanging $x_{111}^2 f_4$ and $d_{12}d_{13}d_{23}$ (these are different, but of the same weight).

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One sees that the multiplicity comes from interchanging $x_{111}^2 f_4$ and $d_{12}d_{13}d_{23}$ (these are different, but of the same weight).

Thus the multiplicity is closely related to our relation $f_3^2 = x_{111}^2 f_4 - 4d_{12}d_{13}d_{23}$.

3 qubits: $GL_2^{\times 3}$-orbits in $(\mathbb{C}^2)^{\otimes 3}$ and entanglement

We denote by π_{ij} the representation generated by d_{ij}, and by ρ_3 the representation generated by f_3. We say that $\pi_{ij} = 0$ on a set A if every element of π_{ij} is 0 on A, and $\pi_{ij} \neq 0$ on A if some element of π_{ij} is $\neq 0$ on A. (Likewise for ρ_3.)

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There are six (nonzero) orbits:

(1) The open orbit is given by $f_4 \neq 0$. It consists of rank two tensors.

(2) The next orbit, which is open in the zero set of f_4, is given by $f_4 = 0$, $\rho_3 \neq 0$. It consists of rank three tensors.

3 qubits: $GL_2^{\times 3}$-orbits in $(\mathbb{C}^2)^{\otimes 3}$ and entanglement

(3) The next three orbits are given by $\pi_{12} = \pi_{13} = 0$, $\pi_{23} \neq 0$, respectively $\pi_{12} = \pi_{23} = 0$, $\pi_{13} \neq 0$ respectively $\pi_{13} = \pi_{23} = 0$, $\pi_{12} \neq 0$. Each of these orbits consists of rank two tensors.

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(4) The orbit of decomposable (rank one) tensors is given by $\pi_{12} = \pi_{13} = \pi_{23} = 0$.

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Notice that the rank is not maximal generically. (It is ≤ 3, and generically 2.)

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We skip the (easy and entertaining) proof.

3 qubits: $U_2^{\times 3}$-invariants in $\mathcal{P}((\mathbb{C}^2)^{\otimes 3})_{\mathbb{R}}$

Following Meyer-Wallach, we consider real polynomials, i.e., polynomials in variables x_{ijk} and $\overline{x}_{ijk}$.

3 qubits: $U_2^{\times 3}$-invariants in $\mathcal{P}((\mathbb{C}^2)^{\otimes 3})_{\mathbb{R}}$

Following Meyer-Wallach, we consider real polynomials, i.e., polynomials in variables x_{ijk} and $\overline{x}_{ijk}$.

The $U_2^{\times 3}$-invariants in the real polynomials separate $U_2^{\times 3}$-orbits in $(\mathbb{C}^2)^{\otimes 3}$, hence can be used to describe a classification of these orbits.

3 qubits: $U_2^{\times 3}$-invariants in $\mathcal{P}((\mathbb{C}^2)^{\otimes 3})_{\mathbb{R}}$

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Moreover, if $V \cong W$, then there is an invariant in $V\overline{W}$, as well as one in $W\overline{V}$. (The latter is the complex conjugate of the former.)

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We denote by $\phi_1, \phi_{12}, \phi_{13}, \phi_{23}, \phi_3$ and ϕ_4 the invariants corresponding respectively to the highest weight vectors $x_{111}, d_{12}, d_{13}, d_{23}, f_3$ and f_4; here each representation gets combined with its own complex conjugate.

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Furthermore, let ζ_6 denote the invariant obtained by combining the representation with highest weight vector $x_{111}^2 f_4$ with the complex conjugate of the (equivalent) representation with highest weight vector $d_{12}d_{13}d_{23}$.

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The invariants $\phi_1, \phi_{12}, \phi_{13}, \phi_{23}, \phi_3, \phi_4, \zeta_6$ and $\overline{\zeta}_6$ generate the algebra of invariants. Their degrees are respectively 2,4,4,4,6,8,12 and 12, and it is easy to write down their leading terms.

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To finish the classification of $U_2^{\times 3}$-orbits, one needs to write down the relations between the above generators, and then assign values to generators so that the relations are satisfied. Also, the values of $\phi_1, \phi_{12}, \phi_{13}, \phi_{23}, \phi_3, \phi_4$ have to be positive, while the value of $\overline{\zeta}_6$ must be complex conjugate to the value of ζ_6.

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These orbits were written down by Brylinski, and we get to reproduce his result with a different proof.

THANK YOU!