BRST and beyond

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January 21, 2025

Contents

Introduction

This talk will be discussing BRST formalism[1] and demonstrating it on two examples: mechanical system with SO(3) symmetry and Yang-Mills theory. Afterwards, we will move on to the BV[2], again we will be using Yang-Mills as an example and finally we will take a look at the homological perturbation lemma (HPL).

[1] C. Becchi, A. Rouet, and R. Stora, "Renormalization of Gauge Theories," Annals Phys., vol. 98, pp. 287–321, 1976.

[2] I. A. Batalin and G. A. Vilkovisky, "Quantization of Gauge Theories with Linearly Dependent Generators," Phys.

BRST Introduction

We start this section by listing some ingredients and either indroducing them or reminding oureselves of them. We will mostly follow book by Marián Fecko[3] and notes by José Figueroa-O'Farrill[4]

[3]Fecko, M. (2006). Differential Geometry and Lie Groups for Physicists.
[4]Figueroa-O'Farrill, J. (2006). BRST Cohomology

Ingredients

What we need:

- symplectic manifold $\mathcal{M} \equiv (\mathcal{M}, \omega)$
- Lie group G and its algebra $\mathcal{G}$
- moment map $\Phi : \mathcal{M} \to \mathcal{G}^*$
- Chevalley-Eilenberg complex
- Koszul resolution
- ghosts and antighosts

$\mathcal{M}$ & G

$\mathcal{M}$:

- a symplectic manifold with symplectic form ω
- the symplectic form – non-degenerate, closed 2-form

G:

- Lie group – a group that is also a smooth manifold
- has a Lie algebra $\mathcal{G}$ – a linear space with bilinear operation $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ that is antisymmetric and satisfies Jacobi identity

Moment map

Moment map ($\Phi : \mathcal{M} \to \mathcal{G}^*$):

- $i_{\zeta_X}\omega = -d\Phi_X$, where i is the interior product, ζ_X is Hamiltonian vector field, ω is symplectic form and Φ_X is a zero form (also sometimes referred to as moment map)
- $\Phi_{X+\lambda Y} = \Phi_X + \lambda\Phi_Y$, where $X, Y \in \mathcal{G}$ and $\lambda \in \mathbb{C}$
- $\{\Phi_X, \Phi_Y\} = \Phi_{[X,Y]}$, where $X, Y \in \mathcal{G}$, $\{-,-\}$ is Poisson bracket defined $\omega(\zeta_f, \zeta_g) := \{f, g\}$ and $[-,-]$ is the Lie bracket of $\mathcal{G}$

Complex and resolution

Chevalley-Eilenberg chain complex $(C^p(\mathcal{G}, V), d_{CE})$:

- vector space $C^p = \Lambda^p \mathcal{G}^* \otimes V$, where V is the representation space of a representation ρ
- differential $d_{CE} = -\frac{1}{2} f_{ij}^k \varepsilon^i \varepsilon^j i_k \otimes \hat{1} + \varepsilon^i \otimes \rho(E_i)$, where $\varepsilon^i \alpha = E^i \wedge \alpha$, i is interior product, f_{ij}^k are structure constants and E_i (E^i) is a base of $\mathcal{G}$ $(\mathcal{G}^*)$
- $d_{CE}^2 = 0$

Koszul resolution:

- Koszul cochain complex (K^p, δ), where $K^p = \Lambda^p \mathcal{G} \otimes C^\infty(\mathcal{M})$
- cohomology of this complex is $H^i(K^\bullet) = \begin{cases} C^\infty(\mathcal{M}_P), \textit{ for } i{=}0 \\ 0, \textit{ for } i{\neq}0 \end{cases}$

Bigraded complex

Now we combine complex and resolution and get

$$C^p(\mathcal{G}, K^q) = \Lambda^p \mathcal{G}^* \otimes \Lambda^q \mathcal{G} \otimes C^\infty(\mathcal{M}) \equiv C^{p,q}, \tag{1}$$

which is Chevalley-Eilenberg complex where the representation space is chosen to be the Koszul complex. The differential for this bicomplex would be $D = d_{CE} + (-1)^p \delta$. However, this differential does not respect the bidegree, only the total degree $n = p - q$ and therefore the complex we will use is $\mathscr{C}^n = \bigoplus_{p-q=n} C^{p,q}$ with $D : \mathscr{C}^n \to \mathscr{C}^{n+1}$. As for the cohomology $H^i(\mathscr{C}^\bullet) \cong H^i(\mathcal{G}, C^\infty(\mathcal{M}_P))$.

In diagrams

The bidegree diagram:

$$\begin{tikzcd} C^p(\mathcal{G}, K^q) \arrow[r, "d_{CE}"] \arrow[d, "\delta"] & C^{p+1}(\mathcal{G}, K^q) \arrow[d, "\delta"] \\ C^p(\mathcal{G}, K^{q-1}) \arrow[r, "d_{CE}"] & C^{p+1}(\mathcal{G}, K^{q-1}) \end{tikzcd} \tag{2}$$

The total degree diagram:

$$\mathscr{C}^n \xrightarrow{D} \mathscr{C}^{n+1}, \tag{3}$$

where $\mathscr{C}^n = C^{n,0} \oplus C^{n+1,1} \oplus \ldots \oplus C^{N-1,N-n-1} \oplus C^{N,N-n}$

Bigraded complex diagram

$$
\begin{array}{ccccccccc}
C^{0,0} & \xrightarrow{d} & C^{1,0} & \xrightarrow{d} & C^{2,0} & \xrightarrow{d} & C^{3,0} & \xrightarrow{d} & \dots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^{0,1} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & C^{2,1} & \xrightarrow{d} & C^{3,1} & \xrightarrow{d} & \dots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^{0,2} & \xrightarrow{d} & C^{1,2} & \xrightarrow{d} & C^{2,2} & \xrightarrow{d} & C^{3,2} & \xrightarrow{d} & \dots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^{0,3} & \xrightarrow{d} & C^{1,3} & \xrightarrow{d} & C^{2,3} & \xrightarrow{d} & C^{3,3} & \xrightarrow{d} & \dots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\dots & & \dots & & \dots & & \dots & &
\end{array}
$$

(For clarity of the diagram on this and the next slide $d \equiv d_{CE}$)

Bigraded complex diagram

$$
\begin{array}{ccccccccc}
C^{0,0} & \xrightarrow{d} & C^{1,0} & \longrightarrow & C^{2,0} & \longrightarrow & C^{3,0} & \longrightarrow & \dots \\
\delta\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^{0,1} & \longrightarrow & C^{1,1} & \longrightarrow & C^{2,1} & \longrightarrow & C^{3,1} & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^{0,2} & \longrightarrow & C^{1,2} & \longrightarrow & C^{2,2} & \longrightarrow & C^{3,2} & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^{0,3} & \longrightarrow & C^{1,3} & \longrightarrow & C^{2,3} & \longrightarrow & C^{3,3} & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & & \dots & & \dots & & \dots & &
\end{array}
$$

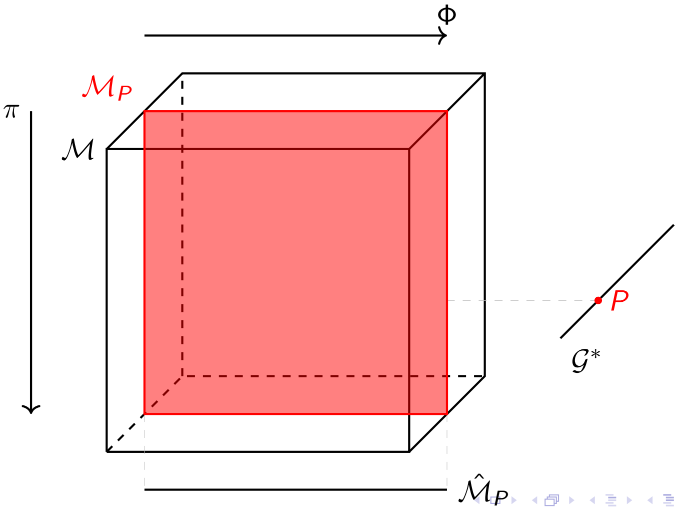
$$\mathscr{C}^0 = C^{0,0} \oplus C^{1,1} \oplus C^{2,2} \oplus \dots \tag{4}$$

Symplectic reduction

Symplectic reduction:

- takes the symplectic manifold $\mathcal{M}$ and reduces it to a smaller symplectic manifold $\hat{\mathcal{M}}_P$
- projection $\pi : \mathcal{M}_P \to \hat{\mathcal{M}}_P$
- $\mathcal{M}_P := \{m \in \mathcal{M} | \Phi(m) = P\}$, where $P \in \mathcal{G}^*$ and $\Phi : \mathcal{M} \to \mathcal{G}^*$ is the moment map
- symplectic form: $\pi^* \hat{\omega} = j^* \omega$, where $j : \mathcal{M}_P \to \mathcal{M}$ is an inclusion
- (notation often is $\hat{\mathcal{M}}_P \equiv \mathcal{M}//G$)

Symplectic reduction



BRST

Our complex $(\mathscr{C}^n, Q_{BRST})$ consists of:

- $C^{p,q} = \Lambda^p \mathcal{G}^* \otimes \Lambda^q \mathcal{G} \otimes C^\infty(\mathcal{M})$
- $Q_{BRST} = \{d_{CE}, -\}$, where d_{CE} in the ghost notation is $d_{CE} = c^i \Phi_i - \frac{1}{2} f_{ij}^k c^i c^j b_k$

The ghost notation:

- ghost c, Grassman odd variable, $gh(c) = 1$ (ghost number)
- antighost b, Grassman odd variable, $gh(b) = -1$
- $\{c^i, b_j\} = \delta^i_j$, $\{c, c\} = 0 = \{b, b\}$

Cohomology of BRST

Difficult and long algebra provides an isomorphism[5]:

$$H^0(\mathcal{G}, C^\infty(\mathcal{M}_P)) \cong C^\infty(\hat{\mathcal{M}}_P), \tag{5}$$

$$H^0(\mathscr{C}^\bullet) \cong C^\infty(\hat{\mathcal{M}}_P), \tag{6}$$

which tells us that instead of dealing with the algebraic problem of smooth functions on reduced symplectic manifold, we can simply focus on cohomologies of the BRST complex.

[5] This is explicitly derived in the notes of Figueroa-O'Farrill (Figueroa-O'Farrill, J. (2006). BRST Cohomology)

SO(3) example

Let us have a mechanical system with SO(3) symmetry.[6]

- structure constants $f_{ij}^{k} = \epsilon_{ij}^{k}$
- moment map is the angular momentum $\Phi_i = L_i = \epsilon_{ij}^{k} x^j p_k$
- coordinates are orthogonal to moment map $\vec{x} \cdot \vec{L} = 0$
- $\{L_i, L_j\} = \epsilon_{ij}^{k} L_k \sim c_{ij} x^k L_k$ – structure constants are not unique
- solution is new ghost/antighost pair and additional term in the differential

[6] Hancharuk, A and Strobl, T. (2022) BFV extensions for mechanical systems with Lie-2 symmetry.

SO(3) example

The new ghost pair:

- ghost η, Grassman odd variable, $gh(\eta) = 2$
- antighost ρ, Grassman odd variable, $gh(\rho) = -2$

The new differential will be $Q_{BRST} = \{d_{new}, -\}$, where

$$d_{new} = c^i L_i - \frac{1}{2}\epsilon_{ij}^k c^i c^j b_k + \eta x^i b_j \tag{7}$$

Yang-Mills example

The Yang-Mills theory consists of:

- Action $S_{YM} = -\frac{1}{2}\int F \wedge *F$
- Field strength $F = dA + \frac{1}{2}[A \wedge A]$
- Covariant derivative $D_A = d + [A, -]$
- Potential $A \to A + D_A \lambda$

To get BRST we switch function λ with a ghost field c. This will also introduce the antighost field b and the Nakanishi-Lautrup field h

Yang-Mills example

The differential will act as:

- $Q_{BRST}c = -\frac{1}{2}[c, c]$
- $Q_{BRST}b = h$
- $Q_{BRST}h = 0$
- $Q_{BRST}A = D_A c$

The Yang-Mills action

- BRST-invariant $Q_{BRST}S_{YM} = 0$
- we have to add term $Q_{BRST}(bd^*A) = hd^*A + bd^*D_A c$
- action $S = S_{YM} + \int(hd^*A + bd^*D_A c)$
- partition function $Z = \int \mathcal{D}A\mathcal{D}c\mathcal{D}b\mathcal{D}h \; e^{iS_{YM}} e^{i\int(hd^*A + bd^*D_A c)}$

BV and HPL introduction

In this section we are going to show antifields and then take a look at action integral of Yang-Mills theory. In the last part we will talk about homotopy equivalence, deformation retract and HPL. We will follow shortened version of master thesis by Ján Pulmann[7]

[7] Pulmann, J. (2016). Effective action and homological perturbation lemma

Antifields

Let us start with the classification of the fields. We have fields that are Grassman odd/even, and we have ghosts/antighosts. To only have one grading let statistic of a field χ be $gh(\chi)\ mod 2$.

Antifields:

- antifield of χ will be $\chi^{\ddagger}$
- opposite statistics as its field
- $gh(\chi) + gh(\chi^{\ddagger}) = -1$

Statistics

The relation $gh(\chi) + gh(\chi^{\ddagger}) = -1$ gives $gh(c^{\ddagger}) = -2$, $gh(b^{\ddagger}) = 0$ and $gh(h^{\ddagger}) = -1$.

Grassman parity :	even	odd	even	odd
$gh(-)$:	-2	-1	0	1
fields	$c^{\ddagger}$	$A^{\ddagger}, b, h^{\ddagger}$	$A, h, b^{\ddagger}$	c

Yang-Mills

Starting from BRST of Yang-Mills action:

- $S[A, c, b, h] = S_{YM}[A] + Q_{BRST}\Psi[A, c, b, h]$, where $\Psi = \int bd^*A$
- using antifields[8]: $S[A, \chi, \chi^\ddagger] = S_{YM}[A] + (Q_{BRST}\chi)^n \chi^\ddagger_n$
- $\chi^\ddagger$ is given by $\chi^\ddagger_n = \frac{\delta\Psi}{\delta\chi^n}$
- classical master equation: $\frac{\delta_R S}{\delta\chi^\ddagger_n}\frac{\delta_L S}{\delta\chi^n} = 0$

[8]For simplicity we will omit the integral in the last term but will keep in mind that it is there: $\int (Q_{BRST}\chi)^i \chi^\ddagger_i \equiv (Q_{BRST}\chi)^n \chi^\ddagger_n$

Quantum master equation

Quantum master equation:

- we generalize action so that it is no longer linear in antifields
- $\Delta e^{\alpha S[\chi,\chi^\ddagger]} = 0 \implies \{S, S\} + \frac{2}{\alpha}\Delta S = 0$
- α is a parameter that is usually chosen to be $-\frac{1}{\hbar}$
- bracket $\{A, B\} = \frac{\delta_R A}{\delta \chi^n}\frac{\delta_L B}{\delta \chi_n^\ddagger} - \frac{\delta_R A}{\delta \chi_n^\ddagger}\frac{\delta_L B}{\delta \chi^n}$
- BV Laplacian $\Delta A = (-1)^{|A||\chi^n|+1}\frac{\delta_L}{\delta \chi^n}\frac{\delta_R A}{\delta \chi_n^\ddagger}$
- for classical part of action: $\{S_{\text{class}}, S_{\text{class}}\} = 0$

Homotopy equivalence

The homotopy equivalence is given by:

$$\begin{tikzcd} (V, d_V) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (W, d_W) \arrow[l, shift left, "i"] \end{tikzcd} \tag{8}$$

- (V, d_V) and (W, d_W) are chain complexes
- p, i are quasi-isomorphisms, $i \circ p = 1 + d_V \circ h + h \circ d_V$
- for $p \circ i = \mathbb{1}$ on W – deformation retract
- if also $h \circ i = 0$, $p \circ h = 0$ and $h \circ h = 0$ – special deformation retract

Homological perturbation lemma

HPL: Let us have homotopy equivalence (8) and $\delta : V \to V$ such that $|d_V| = |\delta|$ and $(d_V + \delta)^2 = 0$, then for invertible $(1 - \delta h)$ there is a homotopy equivalence:

$$\begin{tikzcd} (V, d_V + \delta) \arrow[loop left, "h'"] \arrow[r, shift left, "p'"] & (W, d'_W) \arrow[l, shift left, "i'"] \end{tikzcd} \tag{9}$$

and if the original homotopy equivalence was a special deformation retract, this perturbed one is also.

HPL

$$
\begin{tikzcd}
(V, d_V + \delta) \arrow[loop left, "h'"] \arrow[r, shift left, "p'"] & (W, d'_W) \arrow[l, shift left, "i'"]
\end{tikzcd} \tag{10}
$$

$$h' = h + hA\delta h \tag{11}$$

$$h' = p + pA\delta h \tag{12}$$

$$i' = i + hA\delta i \tag{13}$$

$$d'_W = d_W + pA\delta i \tag{14}$$

where $A = (1 - \delta h)^{-1}$.

References

1. Fecko, M. (2006). Differential geometry and Lie groups for physicists. (book)
2. Figueroa-O'Farrill, J. (2006). BRST Cohomology (notes available online)
3. Boffo, E. (2024) TQFT course (notes available in my notebooks)
4. Hancharuk, A and Strobl, T. (2022) BFV extensions for mechanical systems with Lie-2 symmetry.
5. Pulmann, J. (2016). Effective action and homological perturbation lemma