On the Universal Drinfeld–Yetter algebra

Andrea Rivezzi

45th Winter School Geometry and Physics, Srni
Based on arXiv:2404.16786

Motivation: universal quantization functors

(1992): V. Drinfeld announces several problems in quantum group theory. Among them, there is the problem of finding a universal quantization of Lie bialgebras.

(1998): P. Etingof and D. Kazhdan construct a quantization technique for Lie bialgebras (1996), and then they provide a universal formulation (1998).

(2019): Following an idea of B. Enriquez (2001–2005), A. Appel and V. Toledano Laredo define a cochain complex of algebras $\{\mathfrak{U}^n\}_{n\geqslant 0}$ whose cohomology controls the existence of universal quantization functors of Lie bialgebras.

The aim of this talk is to present the structure and the combinatorial properties of the algebra $\mathfrak{U} := \mathfrak{U}^1$.

PROPs and colored PROPs

- A **PROP** (product and permutation category) is a linear, symmetric strict monoidal category whose objects are indexed by the natural numbers $\{[0], [1], [2], \ldots\}$ and whose tensor product is given by $[n] \otimes [m] = [n+m]$ (the tensor unit is $[0]$).
- A **colored PROP** is a linear, strict, symmetric monoidal category whose objects are finite sequences over a set A. Here the tensor product of two elements is the concatenation of sequences, and the unit with respect to the tensor product is the empty sequence.
- Let P, Q be two PROPs. A **universal construction** from P to Q is a monoidal functor $F : \mathsf{Q} \to \mathsf{P}$.

PROPs are commonly used to provide a universal framework for the description of certain categories of algebraic objects.

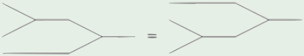
Pictorial representation

PROPs are usually described using a pictorial representation:

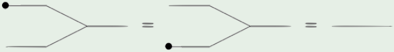
Identity on the n-th object Permutations Products Coproducts

Example

The PROP AA of associative algebras is the one generated by two morphisms  and relations



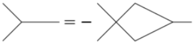
and



The PROP of Lie bialgebras

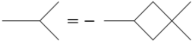
The PROP of Lie bialgebras is the one generated by two morphisms

and by relations

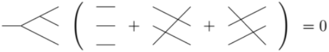


antisymmetry of the bracket

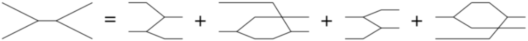
Jacobi rule



antisymmetry of the cobracket



coJacobi rule

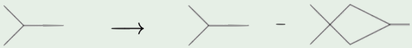


Cocycle identity

Universal constructions

Example

Let AA and LA be the PROPs of associative algebras and of Lie algebras. The universal construction $F : \mathrm{LA} \to \mathrm{AA}$ determined by



is the PROPic version of the usual commutator functor.

Theorem (P. Etingof, D. Kazhdan, 1998)

There is a universal construction $Q : \mathrm{QUE} \to \underline{\mathrm{LBA}}[[\hbar]]$ *such that*

$$Q(m) = m_0 \mod \hbar$$
$$Q(\Delta) = \Delta_0 \mod \hbar$$
$$Q(\Delta - (12) \circ \Delta) = \hbar\delta \mod \hbar^2$$

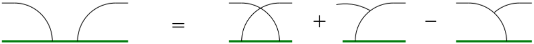
The colored PROP of Drinfeld–Yetter modules

The PROP of Drinfeld–Yetter modules is the colored PROP DY generated by two objects $[\mathfrak{b}]$ and $[V]$, four morphisms

and by the relations of the PROP of Lie bialgebras, together with the following relations

Compatibility between the action and the Lie bracket

Compatibility between the coaction and the Lie cobracket



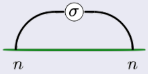
Compatibility between the action and the coaction

The universal Drinfeld–Yetter algebra

The **universal Drinfeld–Yetter algebra** is $\mathfrak{U} = \mathrm{End}_{\mathrm{DY}}([V])$.

Theorem (A. Appel, V. Toledano Laredo, 2019)

For any $n \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_n$ denote by r_n^σ the element of $\mathfrak{U}$ pictorially represented by



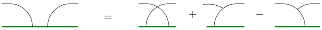
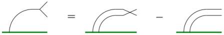
Then the collection $\mathcal{B} = \{r_n^\sigma, n \geqslant 0, \sigma \in \mathfrak{S}_n\}$ is a basis of $\mathfrak{U}$.

Hence, we have an isomorphism of vector spaces

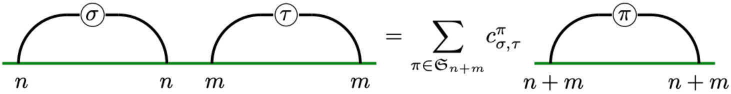
$$\mathfrak{U} \cong \bigoplus_{n \in \mathbb{N}} \mathbb{K}[\mathfrak{S}_n]$$

Algebra structure of $\mathfrak{U}$

One can compute the multiplication $r_n^\sigma \circ r_m^\tau$ through the following algorithm:

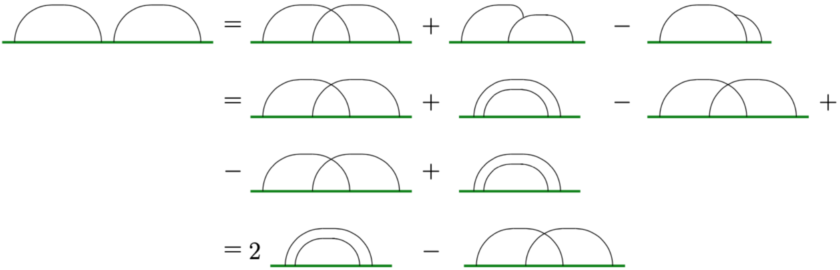
- Apply the identity  until no action precede any coaction.
- Apply the identity  until no Lie bracket appears in the expansion.
- Apply the identity  until no Lie cobracket appears in the expansion.

Hence $\mathfrak{U}$ carries a $\mathbb{N}$–graded algebra structure with integral coefficients $c_{\sigma,\tau}^\pi \in \mathbb{Z}$:



Some examples

The simplest example: $r_1^{\mathrm{id}} \circ r_1^{\mathrm{id}} = 2r_2^{\mathrm{id}} - r_2^{(12)}$:



Proposition

We have $r_n^{\mathrm{id}} \circ r_1^{\mathrm{id}} = (n+1)r_{n+1}^{\mathrm{id}} - \sum_{i=1}^{n} r_{n+1}^{(i,i+1)}$.

However, the number of terms appearing after the application of the algorithm is quite big:

	1	**2**	**3**	**4**
1	5	17	53	161
2	17	129	785	4353
3	53	785	8165	72353
4	161	4353	72353	958977

Conjecture

The number of summands of $r_n^\sigma \circ r_m^\tau$ is

$$\sum_{k=0}^{m} \sum_{i=0}^{k} (-1)^{m-k} \binom{k}{i} (2i+1)^m (2k+1)^n$$

The problem of finding a formula for the coefficients $c^{\pi}_{\sigma,\tau}$ in terms of symmetric groups is open.

Drinfeld–Yetter Looms and an explicit formula

In arXiv:2404.16786 I defined a family of combinatorial objects $\mathfrak{L}_{n,m}$, called **Drinfeld–Yetter Looms**, with the following properties:

- The set $\mathfrak{L}_{n,m}$ is defined as a set of fillings of an empty grid with n rows and m columns by some specific tiles and according to some specific rules.
- To any element $L \in \mathfrak{L}_{n,m}$ we can associate a sign, depending on the tiles occurring in L.
- To any triple of permutations σ, τ, π we can associate a set of looms $\mathfrak{L}_{n,m}^{\sigma,\tau,\pi}$.
- For any $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_m$ we have

  $$c_{\sigma,\tau}^{\pi} = P_{n,m}^{\sigma,\tau,\pi} - N_{n,m}^{\sigma,\tau,\pi}.$$

  where $P_{n,m}^{\sigma,\tau,\pi}$ (resp. $N_{n,m}^{\sigma,\tau,\pi}$) denotes the number of positive (resp. negative) Drinfeld–Yetter Looms *associated to the permiutations* σ, τ, π.
- Drinfeld–Yetter Looms have interesting combinatorial properties, such as relationship with Stirling numbers, bumpless pipedreams, and permutation patterns.

Example

Let us compute $r_1^{\text{id}} \circ r_1^{\text{id}}$. The set of 1×1 Drinfeld–Yetter looms is

$$\mathfrak{L}_{1,1} = \{ \ldots \}.$$

We have

$$P_{1,1}^{\text{id}_1,\text{id}_1,\text{id}_2} = 2 \qquad N_{1,1}^{\text{id}_1,\text{id}_1,\text{id}_2} = 0$$
$$P_{1,1}^{\text{id}_1,\text{id}_1,(12)} = 1 \qquad N_{1,1}^{\text{id}_1,\text{id}_1,(12)} = 2$$

And so

$$c_{\text{id}_1,\text{id}_1}^{\text{id}_2} = 2 \qquad \textit{and} \qquad c_{\text{id}_1,\text{id}_1}^{(12)} = -1.$$

- V. G. Drinfeld, On some unsolved problems in quantum group theory, Quantum groups. proceedings of workshops held in the Euler International Mathematical Institute, Leningrad, fall 1990, 1992, pp. 1-8.
- P. Etingof and D. Kazhdan, Quantization of Lie bialgebras. II, Sel. Math., New Ser. 4 (1998), no. 2, 213-231 (1998).
- B. Enriquez, A cohomological construction of quantization functors of Lie bialgebras, Adv. Math. 197 (2005), no. 2, 430-479.
- A. Appel and V. Toledano Laredo, Uniqueness of Coxeter structures on Kac-Moody algebras, Adv. Math. 347 (2019), 1-104.
- A. Rivezzi, Structure of the universal Drinfeld-Yetter algebra, Preprint arXiv:2404.16786.

Thank you for your attention