On the incompleteness of the classification of quadratically integrable Hamiltonian systems in the three–dimensional Euclidean space

Libor Šnobl

Czech Technical University in Prague

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Introduction

In their seminal paper A systematic search for nonrelativistic systems with dynamical symmetries. Nuovo Cimento A Series 10, 52:1061–1084, 1967 Makarov, Smorodinsky, Valiev and Winternitz presented a list of quadratically integrable natural Hamiltonian systems in 3D Euclidean space and identified them with systems separable in orthogonal coordinate systems. Their result is one of the standard references in the theory of integrable and superintegrable systems. It was widely accepted as a proof of the equivalence of quadratic integrability and separability in Euclidean 3D space.

Introduction

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Review of the original argument and its loophole

Let us review the original argument and indicate the point where the analysis becomes incomplete. We consider the natural Hamiltonian for a particle of unit mass moving in 3D Euclidean space under the influence of the potential $V(\vec{x})$,

$$H = \frac{1}{2}\vec{p}^2 + V(\vec{x}), \quad (1)$$

and assume that it is integrable with a pair of integrals of motion X_1, X_2 which are quadratic polynomials in the momenta, with coordinate dependent coefficients.

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and assume that it is integrable with a pair of integrals of motion X_1, X_2 which are quadratic polynomials in the momenta, with coordinate dependent coefficients. For the sake of simplicity we shall proceed classically; however, the determining equations for the quadratic integrals and their solution are exactly the same in quantum mechanics, assuming total symmetrization of any terms involving noncommuting operators $\hat{x}_a, \hat{p}_a$.

Review of the original argument and its loophole

Notation: the position vector in the Cartesian coordinates is expressed as $\vec{x} = (x, y, z)$, the canonically conjugated momenta to x, y, z are denoted by p_x, p_y, p_z and the angular momenta are expressed as $l_x = yp_z - zp_y$, $l_y = zp_y - xp_z$ and $l_z = xp_y - yp_x$.

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As a consequence of the assumed form of the integrals, the Poisson brackets $\{H, X_1\}_{P.B}$, $\{H, X_2\}_{P.B}$ and $\{X_1, X_2\}_{P.B}$ are third order polynomials in the momenta p_x, p_y, p_z. As the Hamiltonian (1) is an even polynomial in the momenta, the odd and even order terms in the integrals commute with the Hamiltonian (1) independently and the integrals can be assumed to be even or odd polynomials in the momenta. As any first order integral implies the existence of a second order integral as its square, we can assume that X_1 and X_2 are second order even polynomials in the momenta.

Review of the original argument and its loophole

The left hand sides of the involutivity conditions

$$\{H, X_1\}_{P.B} = 0, \quad \{H, X_2\}_{P.B} = 0, \tag{2}$$
$$\{X_1, X_2\}_{P.B} = 0 \tag{3}$$

then become third order odd polynomials in the momenta. As the momenta are arbitrary, all their coefficients must vanish.

Review of the original argument and its loophole

The third order terms in (2–3) are easily solved and imply that the second order terms in X_1 and X_2 must be commuting elements in the universal enveloping algebra of the Euclidean algebra $\mathfrak{U}(\mathfrak{e}_3)$, i.e. quadratic polynomials in the linear and angular momenta. As we may arbitrarily combine the integrals with the Hamiltonian and among themselves, and the systems related by Euclidean transformations are physically equivalent, the leading order terms must belong to any of the classes of three–dimensional Abelian subalgebras consisting of quadratic elements in the universal enveloping algebra of the Euclidean algebra $\mathfrak{U}(\mathfrak{e}_3)$, which were recently classified in A Marchesiello and L Šnobl J. Phys. A 55 (2022) 145203:

Review of the original argument and its loophole

(a) $X_1 = l_1^2 + l_2^2 + l_3^2 + a l_3 p_3 + b p_3^2, \quad X_2 = l_3^2,$

(b) $X_1 = l_1^2 + l_2^2 + l_3^2 + b(a p_2^2 + p_3^2), \quad X_2 = a l_2^2 + l_3^2 - ab p_1^2,$

(c) $X_1 = l_1^2 + l_2^2 + l_3^2 + 2b(l_1 p_1 - (3a-1) l_2 p_2 - 2 l_3 p_3) + 3b^2((1-4a)p_1^2 - (3a^2 - 2a - 1)p_2^2 + 2(a-1)p_3^2), \quad X_2 = a l_2^2 + l_3^2 + 6ab l_1 p_1 + 9ab^2(a p_3^2 + p_2^2),$

(d) $X_1 = l_3^2, \quad X_2 = \frac{1}{2}(l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) + a l_3 p_3,$

(e) $X_1 = l_3^2 + 2a(l_1 p_1 - l_2 p_2) + a^2 p_3^2, \quad X_2 = \frac{1}{2}(l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) - a p_1 p_2,$

(f) $X_1 = l_3^2 + a l_3 p_3 + b p_1^2 + c p_1 p_3 + d p_2 p_3, \quad X_2 = p_3^2,$

(g) $X_1 = l_3^2 + a p_3^2, \quad X_2 = l_3 p_3 + b p_3^2,$

(h) $X_1 = l_1 p_1 + a l_2 p_2 - (a+1) l_3 p_3 + b p_2^2, \quad X_2 = p_1^2 + \frac{2a+1}{a+2} p_2^2,$

(i) $X_1 = l_1 p_1 + a p_2^2 + b p_2 p_3, \quad X_2 = p_1^2,$

(j) $X_1 = l_1 p_1 + a l_2 p_2 - (a+1) l_3 p_3 + \frac{\omega}{2}(l_1 p_3 + p_3 l_1 - l_3 p_1 - p_1 l_3) + 2b p_1 p_2 + c\left(p_2^2 - p_3^2\right), X_2 = p_1^2 + \frac{6\omega}{4a-1} p_1 p_3 + \frac{a+2}{4a-1} p_2^2 - \frac{5a+1}{4a-1} p_3^2,$

(k) $X_1 = p_1^2 + a p_2^2, \quad X_2 = p_2^2 + b p_1 p_2 + c p_1 p_3 + d p_2 p_3.$

Review of the original argument and its loophole

It remains to solve the remaining conditions, which come from linear terms in the momenta in (2–3), namely to determine the scalar terms in the integrals X_1 and X_2, denoted by $m_1(\vec{x})$ and $m_2(\vec{x})$ below, and find the restrictions on the potential $V(\vec{x})$ implied by their existence. The conditions coming from (2) are easily solved with respect to the first order derivatives of $m_1(\vec{x})$ and $m_2(\vec{x})$. Substituting these into (3), one arrives at a set of three equations which are homogeneous linear first order PDEs for the potential $V(\vec{x})$, cf. (7) below. As the coefficients of $\partial_a V$, $a = x, y, z$ form an antisymmetric 3×3 matrix R, it either has rank 2 or vanishes identically.

Review of the original argument and its loophole

At this point the Makarov et al. stated "Thus the potential V either satisfies three first-order equations – a case which will be considered separately – or the consistency conditions (39) are satisfied trivially." and proceeded assuming that the condition (3) vanishes identically. Only under this assumption they arrived at their list of quadratically integrable natural Hamiltonian systems and showed that one by one they precisely match with the separable systems which were classified by Eisenhart in 1930s (separable in the sense that the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, q_3, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \frac{\partial S}{\partial q_3}\right) = 0 \qquad (4)$$

allows a complete additive separation of variables $S(q_i, t) = -Et + \sum_i S_i(q_i)$ in the respective coordinate system (q_1, q_2, q_3)):

Review of the original argument and its loophole

I. Cartesian $X_1 = p_1^2, \quad X_2 = p_2^2.$

II. Cylindrical $X_1 = l_3^2, \quad X_2 = p_3^2.$

III. Elliptic cylindrical $X_1 = l_3^2 + Ap_1^2, \quad X_2 = p_3^2.$

IV. Parabolic cylindrical $X_1 = l_3 p_1 + p_1 l_3, \quad X_2 = p_3^2.$

V. Spherical $X_1 = l_1^2 + l_2^2 + l_3^2, \quad X_2 = l_3^2.$

VI. Prolate spheroidal $X_1 = l_1^2 + l_2^2 + l_3^2 - A(p_1^2 + p_2^2), \quad X_2 = l_3^2.$

VII. Oblate spheroidal $X_1 = l_1^2 + l_2^2 + l_3^2 + A(p_1^2 + p_2^2), \quad X_2 = l_3^2.$

VIII. Parabolic rotational (also known as circular parabolic)
$X_1 = l_3^2, \quad X_2 = l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2,$

IX. Conical $X_1 = l_1^2 + l_2^2 + l_3^2, \quad X_2 = B^2 l_2^2 + C^2 l_3^2.$

X. Ellipsoidal $X_1 = l_1^2 + l_2^2 + l_3^2 + (A^2 + B^2)p_1^2 + A^2 p_2^2 + B^2 p_3^2,$
$X_2 = B^2 l_2^2 + A^2 l_3^2 + A^2 B^2 p_1^2.$

XI. Paraboloidal $X_1 = l_3^2 + A\,(l_1 p_2 + p_2 l_1) - B\,(l_2 p_1 + p_1 l_2) - ABp_3^2,$
$X_2 = (l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) - A(p_2^2 + p_3^2) - B(p_1^2 + p_3^2).$

Review of the original argument and its loophole

Makarov et al. left several problems to be resolved in Part II of their paper and we can assume that they also intended to address the case of the matrix R of rank 2. However, due to external influence of political nature (military occupation of Czechoslovakia by the forces of Soviet Union and its satellites in 1968 and subsequent emigration of P. Winternitz to the other side of Iron Curtain) the Part II was never written.

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The long forgotten assumption on the rank of the matrix R came back to light recently, when we discussed with P. Winternitz the modification of the classification of quadratically integrable systems when linear terms in the momenta are present in the Hamiltonian. I started to address the problem of $\mathrm{rank}\, R = 2$ from the perspective of algebraic classification of leading order terms and arrived at the conclusion that a quadratically integrable nonseparable system does exist.

Quadratically integrable nonseparable system

Let us look for quadratically integrable Hamiltonian systems of the form corresponding to the class (c) above, namely with the integrals of motion of the form

$$\begin{aligned} X_1 &= l_x^2 + l_y^2 + l_z^2 + 2b(l_x p_x - (3a-1) l_y p_y - 2 l_z p_z) + \\ &\quad +3b^2((1-4a)p_x^2 - (3a^2-2a-1)p_y^2 + 2(a-1)p_z^2) + m_1(\vec{x}), \\ X_2 &= a l_y^2 + l_z^2 + 6ab l_x p_x + 9ab^2(a p_z^2 + p_y^2) + m_2(\vec{x}), \\ &\quad 0 < a \leq \frac{1}{2}, b \neq 0. \end{aligned} \tag{5}$$

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X_2 &= a l_y^2 + l_z^2 + 6ab l_x p_x + 9ab^2(a p_z^2 + p_y^2) + m_2(\vec{x}), \\
&\quad 0 < a \leq \frac{1}{2}, b \neq 0.
\end{aligned} \tag{5}
$$

This form of the integrals represents one of the three possibilities for a pair of integrals both involving terms quadratic in angular momenta, the other two being classes (a) and (b) where

Quadratically integrable nonseparable system

the class (a) includes pairs of integrals of motion arising from spherical, oblate and prolate spheroidal separation of variables, class (b) corresponds to integrals of the systems separable in conical or ellipsoidal coordinates. The assumed form of the integrals (5) is not equivalent to any other known one, e.g. to the ones known from Makarov et al.

Quadratically integrable nonseparable system

As a consequence of the assumed form of the integrals, i.e. the leading order terms of (5) forming an Abelian subalgebra of $\mathfrak{U}(\mathfrak{e}_3)$, the Poisson brackets $\{H, X_1\}_{P.B}$, $\{H, X_2\}_{P.B}$ and $\{X_1, X_2\}_{P.B}$ reduce to first order polynomials in the momenta p_x, p_y, p_z, without zeroth order terms. Separating the conditions (2) into coefficients of p_x, p_y, p_z and solving them with respect to the first order derivatives of $m_1(\vec{x})$ and $m_2(\vec{x})$ we find:

Quadratically integrable nonseparable system

$$
\begin{aligned}
\partial_x m_1(\vec{x}) &= 2(3(1-4a)b^2 + y^2 + z^2)\partial_x V(\vec{x}) - 2(3abz + xy)\partial_y V(\vec{x}) \\
&\quad +2(3by - xz)\partial_z V(\vec{x}), \\
\partial_y m_1(\vec{x}) &= -2(3abz + xy)\partial_x V(\vec{x}) + 2(3(1+2a-3a^2)b^2 + x^2 + z^2)\partial_y V(\vec{x}) \\
&\quad -2(3b(1-a)x + yz)\partial_z V(\vec{x}), \\
\partial_z m_1(\vec{x}) &= 2(3by - xz)\partial_x V(\vec{x}) - 2(3b(1-a)x + yz)\partial_y V(\vec{x}) \\
&\quad +2(6(a-1)b^2 + x^2 + y^2)\partial_z V(\vec{x}), \\
\partial_x m_2(\vec{x}) &= 2(az^2 + y^2)\partial_x V(\vec{x}) - 2(3abz + xy)\partial_y V(\vec{x}) \\
&\quad +2a(3by - xz)\partial_z V(\vec{x}), \\
\partial_y m_2(\vec{x}) &= -2(3abz + xy)\partial_x V(\vec{x}) + 2(9ab^2 + x^2)\partial_y V(\vec{x}), \\
\partial_z m_2(\vec{x}) &= 2a(3by - xz)\partial_x V(\vec{x}) + 2a(9ab^2 + x^2)\partial_z V(\vec{x}).
\end{aligned} \tag{6}
$$

Their compatibility implies a set of second order linear PDEs for the potential $V(\vec{x})$.

Quadratically integrable nonseparable system

On the other hand, substituting (6) into (3) we obtain a set of three first order linear homogeneous PDEs for the potential $V(\vec{x})$. They can be expressed in the matrix form

$$R \cdot (\partial_x V(\vec{x}), \partial_y V(\vec{x}), \partial_z V(\vec{x}))^T = 0 \tag{7}$$

where the matrix R is antisymmetric, $R + R^T = 0$, and its independent elements read

$$\begin{aligned} R_{12} &= (1-a)azx^2 - 6(1-a)abyx - zy^2a - a^2z^3 - 9a^2b^2(1-a)z, \\ R_{13} &= (1-a)yx^2 + 6(1-a)abzx + y^3 + a(9(a-1)b^2 + z^2)y, \\ R_{23} &= -(1-a)^2x^3 + (1-a)(9a(a-1)b^2 + az^2 - y^2)x - 6(1-a)abyz. \end{aligned} \tag{8}$$

As R does not identically vanish for any choice of the parameters a, b and rank of any antisymmetric matrix is even, we are indeed considering the case with $\operatorname{rank} R = 2$.

Quadratically integrable nonseparable system

Solving (7) using the method of characteristics we find that $V(\vec{x})$ must be an arbitrary function $v(u)$ of an invariant coordinate u, which can be conveniently chosen as

$$\begin{aligned} u &= (a-1)^2x^4 + (az^2+y^2)^2 + 2(1-a)x^2(y^2-az^2) + \\ & 6ab(a-1)\left(3\left((x^2-z^2)a - x^2 + y^2\right)b - 4xyz\right) + 81a^2(1-a)^2b^4. \end{aligned} \tag{9}$$

Substituting $V(\vec{x}) = v(u)$ into the compatibility conditions for (6) we find a system of ODEs which reduces to a single equation

$$2u\frac{\mathrm{d}^2v(u)}{\mathrm{d}u^2} = -3\frac{\mathrm{d}v(u)}{\mathrm{d}u}, \tag{10}$$

thus up to an irrelevant additive constant we have the potential

$$V(\vec{x}) = v(u) = \frac{w_0}{\sqrt{u}}. \tag{11}$$

Quadratically integrable nonseparable system

In the next step, the equations (6) determine the scalar terms m_1 and m_2 in the integrals up to irrelevant additive constants. They read

$$\begin{aligned} m_1(\vec{x}) &= 2w_0 \frac{x^2+y^2+z^2+3b^2(1-a)}{\sqrt{u}}, \\ m_2(\vec{x}) &= w_0 \frac{x^2+y^2+a(x^2+z^2)+9ab^2(a+1)}{\sqrt{u}}, \end{aligned} \tag{12}$$

where u is the quartic polynomial in the coordinates introduced in (9).

Quadratically integrable nonseparable system

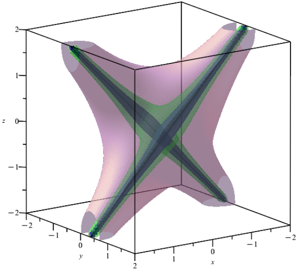
Let us mention that the polynomial (9) can of course have real roots and thus the potential (11) may blow up in the configuration space. We find that $V(\vec{x})$ blows up along the two straight lines given by

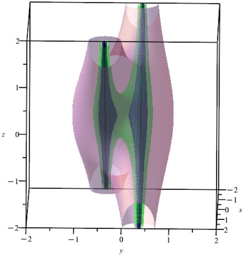
$$x = -\epsilon\sqrt{\frac{a}{1-a}}z, \quad y = 3\epsilon\sqrt{a(1-a)}b, \qquad \epsilon = \pm 1. \qquad (13)$$

As these do not separate $\mathbb{R}^3$ into disconnected domains, everywhere else the potential (11) is a well–defined real function. If we assume that the parameter w_0 is positive, the singular lines (13) are not dynamically accessible for any initial condition with finite energy. Thus our Hamiltonian system is well–defined in the configuration space defined as $\mathbb{R}^3$ without the two singular lines (13). Whether the singularities are dynamically reachable for negative values of w_0 in finite time we don't know yet.

Quadratically integrable nonseparable system

In order to provide a more intuitive understanding of the potential (11) let us present several of its equipotential surfaces (with the parameters $a = \frac{1}{4}$, $b = 1$, $w_0 = 1$ and the energies $V(x, y, z) = 8$, $V(x, y, z) = 4$ and $V(x, y, z) = 1$, viewed from two different directions):





Nonseparability

Looking for another hypothetical integral at most quadratic in the momenta

$$X = \sum_{a,b=x,y,z} K^{ab}(\vec{x}) p_a p_b + m_3(\vec{x}), \qquad K^{ba}(\vec{x}) = K^{ab}(\vec{x}), \quad (14)$$

assuming that $w_0 \neq 0$ and a, b are chosen in the prescribed range, we find that any such integral of motion must be a linear combination of H, X_1 and X_2, i.e. the system (1) with the potential (11) does not possess any other integrals of motion linear or quadratic in the momenta. Thus the considered system cannot be transformed using Euclidean transformations to any of the quadratically integrable and separable systems of Makarov et al.. Therefore, it provides an example of a quadratically integrable yet not separable natural Hamiltonian system.

Nonseparability using another argument

A different argument arriving at the same conclusion uses a theorem due to Eisenhart. In our setting it states that the system described by the Hamiltonian (1) is separable in an orthogonal coordinate system if and only if two Killing tensors K_1 and K_2 (corresponding to the leading order terms of the integrals) exist such that

- $\left\{\sum_{a,b} K_1^{ab} p_a p_b, \sum_{c,d} K_2^{cd} p_c p_d\right\}_{P.B.} = 0,$
- as (1,1)–tensors after lowering one of their indices by the metric, the Killing tensors K_1 and K_2 possess a basis of common eigenforms, i.e. commute as operators on $T^*\mathbb{R}^3$,
- and the following equation holds

$$\mathrm{d}\left(K_k \cdot \mathrm{d}V\right) = \mathrm{d}\left(\sum_{a,b} (K_k)_a{}^b \partial_b V(\vec{x}) dx^a\right) = 0. \qquad (15)$$

Nonseparability using another argument

The Killing tensors K_1 and K_2 corresponding to the integrals X_1 and X_2 of the form (5) are easily found from the coefficients of p_x, p_y, p_z as in equation (14), namely

$$K_1 = \begin{pmatrix} y^2+z^2+3(1-4a)b^2 & -3abz-xy & 3by-xz \\ -3abz-xy & x^2+z^2+3(1+2a-3a^2)b^2 & 3(a-1)bx-yz \\ 3by-xz & 3(a-1)bx-yz & x^2+y^2+6(a-1)b^2 \end{pmatrix} \quad (16)$$

and

$$K_2 = \begin{pmatrix} az^2+y^2 & -3abz-xy & a(3by-xz) \\ -3abz-xy & x^2+9ab^2 & 0 \\ a(3by-xz) & 0 & ax^2+9a^2b^2 \end{pmatrix}. \quad (17)$$

Commutator $[K_1, K_2]$ is proportional to the antisymmetric matrix

$$\begin{pmatrix} 0 & -a\left(-az^3+(1-a)x^2z-y^2z+6(a-1)bxy+9a(a-1)b^2z\right) & (a-1)x^2y-ayz^2-y^3+6a(a-1)bxz \\ \dots & 0 & (a-1)\left(9a(a-1)b^2x-6abyz+(a-1)x\right. \\ \dots & \dots & 0 \end{pmatrix} \quad (18)$$

Nonseparability using another argument

Thus in our allowed range of the parameters $0 < a \leq \frac{1}{2}$ and $b \neq 0$ the two Killing tensors do not commute, i.e. do not possess a basis of common eigenforms. As no quadratic integrals of motion other than X_1, X_2 and H and thus no Killing tensors other than linear combinations of K_1, K_2 and $\mathbf{1}$ (corresponding to the Hamiltonian H itself) are allowed by the potential (11), the system can not separate in any orthogonal coordinate system.

Conclusions – review of investigated classes

We have explicitly demonstrated that the statement on the equivalence of quadratic integrability and separability in 3D Euclidean space does not hold in general, arriving at a new quadratically integrable yet not separable system with the potential (11).

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It is not yet known whether our system (11) is the sole exception to the conclusions of Makarov et al. or whether other quadratically integrable nonseparable systems in Euclidean 3D space do exist.

Conclusions – review of investigated classes

We have explicitly demonstrated that the statement on the equivalence of quadratic integrability and separability in 3D Euclidean space does not hold in general, arriving at a new quadratically integrable yet not separable system with the potential (11).

It is not yet known whether our system (11) is the sole exception to the conclusions of Makarov et al. or whether other quadratically integrable nonseparable systems in Euclidean 3D space do exist. Thus, a complete re–derivation of the list of quadratically integrable systems based on the classification of the leading order terms is currently under way and we expect to report on it in not too distant future.

Thank you for your attention!