Unstraightening of an operad

SRNÍ 2025

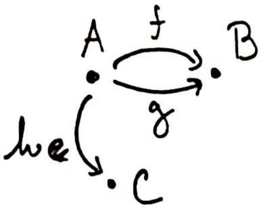
Dominik Trnka

Outline:

- How to integrate a functor.
- How to integrate a monoidal category.
- ~ " ~ categorical operad.

Graph $\Gamma : \cancel{(v \overset{s}{\underset{t}{\leftleftarrows}} e)}\ (e \overset{s}{\underset{t}{\rightrightarrows}} v) \longrightarrow \mathcal{S}et$

$\mathbb{2} \nearrow$



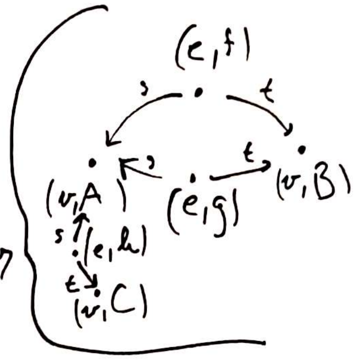
$$E = \Gamma_e \overset{\Gamma_s}{\underset{\Gamma_t}{\rightrightarrows}} \Gamma_v = V$$

— source

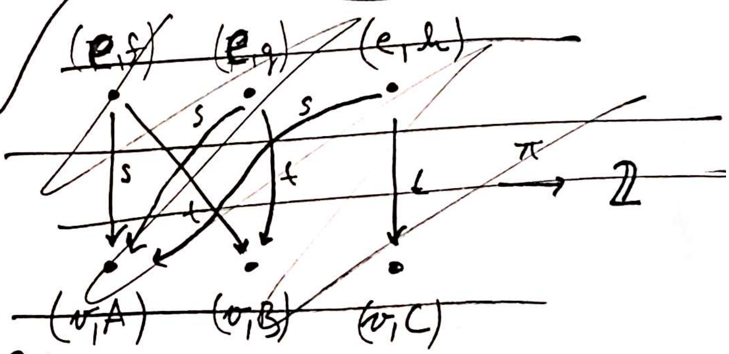
— target

$\Gamma_s(h) = A$

$\Gamma_t(h) = C$



$\int_{\mathbb{2}} \Gamma$ is a category



i) $\int : \text{Graph} \longrightarrow \mathrm{Cat}/_{\mathbb{2}}$ is fully faithful

ii) π is discrete (op)fibration

Myself: $\int : \left\{ \begin{matrix} \text{ns categorical} \\ \text{Operads} \end{matrix} \right\} \xrightarrow{\text{f.f.}} \text{oper. 2-Cat}/_{\Delta}$

Why?

1. more tools to handle Cat-operads
2. represent operads as (operadic) functors $(\Delta \to \widetilde{\mathcal{V}})$
3. if $\mathcal{C} \cong \int \mathcal{P}$, get nice properties (factorisation, universal morphisms)
4. Combine 2 compatible structures & encode new compositional structures

Example 1. $\mathcal{M} \left(= P_1 \,,\, P_{k \neq 1} = \emptyset\right)$

What is $\int \mathcal{M}$?

$\mathcal{M} = (\mathbb{N}, +, \geq)$: $(\mathbb{N}, \geq)$ a poset

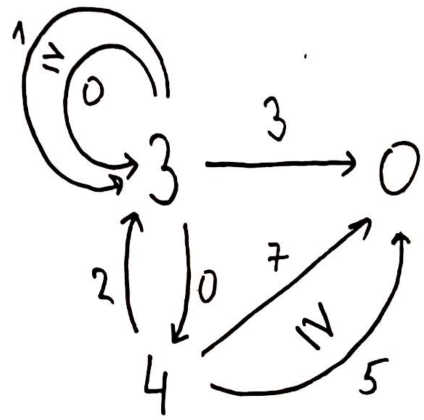
$$\mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N}$$

assoc., unit., pres. $\geq$

$\int \mathbb{N}$ is a 2-category

$$n \xrightarrow[(m+k \geq n)]{k} m$$

$$\begin{tikzcd} n \arrow[r, bend left, "k'"] \arrow[r, bend right, "k''"'] \arrow[r, phantom, "\mathrm{VI}"] & m \end{tikzcd} \quad \text{if } k' \geq k''$$



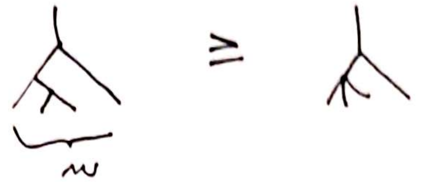
&c.

$\int \mathbb{N}$-structure: $\{A_n\}_{n \in \mathbb{N}}$, $e \in A_0$

$$A_m \times A_k \xrightarrow{\mu} A_n \quad \text{for } m+k \geq n \qquad \left\{ \begin{array}{l} A_n \times A_m \xrightarrow{\mu} A_{n+m} \\ A_m \xrightarrow[\alpha]{} A_n \ \text{ for } m \geq n \end{array} \right.$$

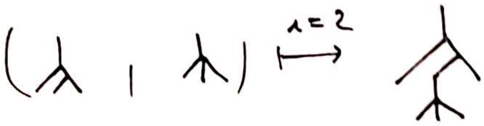
associative, unital & compatible

Example 2.

$\mathcal{T}r_n = \{$  $\} \in \mathrm{Cat}$ (Poset)

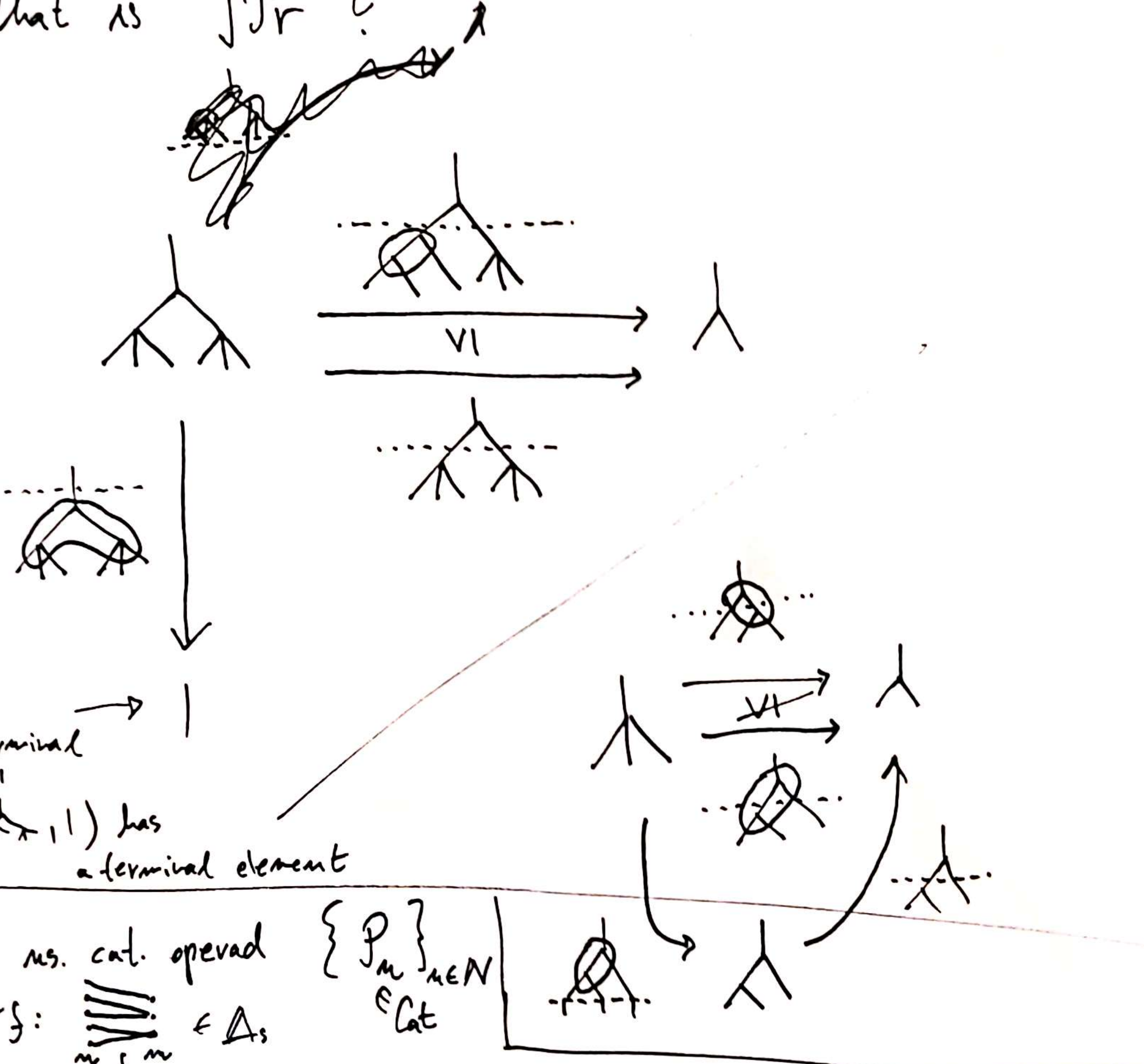
grafting operation

$$\mathcal{T}r_n \times \mathcal{T}r_m \xrightarrow{\circ_i} \mathcal{T}r_{n+m-1}$$



$\Big\}$ ns cat operad (grafting preserves $\geq$)

What is $\int \mathcal{T}r$?



quasi terminal

$\parallel$

$\int \mathcal{T}r(\curlywedge, 1)$ has a terminal element

$\mathcal{P}$ ns. cat. operad $\{\mathcal{P}_n\}_{n\in\mathbb{N}} \in \mathrm{Cat}$

$+\ \forall f: n \xrightarrow{f} m \in \Delta_s$

$$\mathcal{P}_m \times \mathcal{P}_{f^{-1}(1)} \times \cdots \times \mathcal{P}_{f^{-1}(m)} \xrightarrow{\gamma_f} \mathcal{P}_n \quad \text{(assoc \& unital)}$$

functors

$\int \mathcal{P}$ has obj ~~$(n,a)\in\mathcal{P}$~~ $(n, a\in\mathcal{P}_n) \longrightarrow (m, b\in\mathcal{P}_m)$

$$\left(\begin{array}{l} f: n\to m \\ a_i \in \mathcal{P}_{f^{-1}(i)} \\ \gamma_f(b, a_1 \ldots a_m) \xrightarrow{\alpha} a \end{array}\right)$$

& 2cells

Proposition:

i) $\begin{tikzcd} \int \mathcal{P} \arrow[d, "\pi"] \\ \Delta_S \end{tikzcd}$ has a structure of "operadic 2-category"

ii) $\int$ is fully faithful

iii) π is an "operadic fibration".

unary case done:
[2410.05064]
Operadic fibrations & Unary operadic 2-cats.