FACULTY
OF SCIENCE
Masaryk University

Projective invariants of non-torsal ruled surfaces

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Outline

1. Ruled surfaces aka curves in $\mathrm{Gr}_2(4)$
2. Pair of linear second order ODE
3. Osculating rank of the Plücker image
4. Suggested classification

Ruled surface

Definition

$\mathfrak{R}(t) = [a.x(t) + b.y(t)]$ where $x(t) \neq y(t)$ are curves in $\mathbb{R}^4$

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transformations preserving a ruled surface:

- reparametrization $\quad t \mapsto f(s)$
- gauge transformation $\quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \mapsto \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \cdot \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}$

  where $ad - bc \neq 0$

Moving frame of a ruled surface

Ruled surface $\mathfrak{R}(t)$: $t \mapsto [a.x(t) + b.y(t)]$, $\det(x, y, x', y') \neq 0$

Moving frame s: $\mathfrak{R} \to \mathrm{GL}(4, \mathbb{R})$ defined by $t \mapsto g = (x, y, x', y')$

How does g change infinitesimally?

Moving frame of a ruled surface

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How does g change infinitesimally?

$$(x', y', x'', y'') = (x, y, x', y') \cdot \begin{pmatrix} 0 & 0 & q_{11} & q_{21} \\ 0 & 0 & q_{12} & q_{22} \\ 1 & 0 & p_{11} & p_{21} \\ 0 & 1 & p_{12} & p_{22} \end{pmatrix}$$

(Mauer-Cartan form: $s^* \omega = g^{-1} g'$)

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$$\begin{bmatrix} \text{gauge transformation} \\ (x, y \to \text{asymptotic}) \end{bmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & q_{11} & q_{21} \\ 0 & 0 & q_{12} & q_{22} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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$$\begin{bmatrix} \text{reparametrization} \\ + \text{ appropriate gauge} \end{bmatrix} \longrightarrow q_{11} + q_{22} = 0$$

Moving frame of a ruled surface

$$(x', y', x'', y'') = (x, y, x', y') \cdot \begin{pmatrix} 0 & 0 & q_{11} & q_{21} \\ 0 & 0 & q_{12} & -q_{11} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Moving frame of a ruled surface

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zero trace is preserved by

$$t \mapsto \frac{\alpha s + \beta}{\gamma s + \delta} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto B \cdot \begin{pmatrix} w \\ z \end{pmatrix} \qquad (B \in \mathrm{GL}(2) \text{ const.})$$

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$Q = \{q_{ij}\}$ gives complete invariant system up to transformation

$$Q \mapsto f^*(B\,Q\,B^{-1})$$

Pair of linear second order ODE

$$(x', y', x'', y'') = (x, y, x', y') \cdot \begin{pmatrix} 0 & 0 & q_{11} & q_{21} \\ 0 & 0 & q_{12} & q_{22} \\ 1 & 0 & p_{11} & p_{21} \\ 0 & 1 & p_{12} & p_{22} \end{pmatrix}$$

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gives us the second order relation

$$\begin{aligned} x'' &= p_{11}\, x' + p_{12}\, y' + q_{11}\, x + q_{12}\, y \\ y'' &= p_{21}\, x' + p_{22}\, y' + q_{21}\, x + q_{22}\, y \end{aligned}$$

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Set of independent solutions: $\left\{ \begin{pmatrix} x^1 \\ y^1 \end{pmatrix}, \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}, \begin{pmatrix} x^3 \\ y^3 \end{pmatrix}, \begin{pmatrix} x^4 \\ y^4 \end{pmatrix} \right\}$

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Integral ruled surface $\mathfrak{R}(t)$: $t \mapsto [\, a.x(t) + b.y(t)]$

Where $x(t) = [\, x_1(t) : \cdots : x_4(t)],\ y(t) = [\, y_1(t) : \cdots : y_4(t)]$

Pair of linear second order ODE

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Different choice of solutions $\longrightarrow$ projective transformation of $\mathfrak{R}(t)$

Pair of linear second order ODE

$$
\begin{aligned}
x'' &= p_{11}\, x' + p_{12}\, y' + q_{11}\, x + q_{12}\, y \\
y'' &= p_{21}\, x' + p_{22}\, y' + q_{21}\, x + q_{22}\, y
\end{aligned}
$$

Set of independent solutions: $\left\{ \begin{pmatrix} x^1 \\ y^1 \end{pmatrix}, \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}, \begin{pmatrix} x^3 \\ y^3 \end{pmatrix}, \begin{pmatrix} x^4 \\ y^4 \end{pmatrix} \right\}$

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Different choice of solutions $\longrightarrow$ projective transformation of $\mathfrak{R}(t)$

gauge transformation $\longleftarrow$ equivalent pair of ODE

Wilczynski invariants

$$\begin{aligned} x'' &= p_{11}\,x' + p_{12}\,y' + q_{11}\,x + q_{12}\,y \\ y'' &= p_{21}\,x' + p_{22}\,y' + q_{21}\,x + q_{22}\,y \end{aligned} \longrightarrow$$

Wilczynski complete relative invariants [1]

$\theta_4, \theta_{4.1}, \theta_6, \theta_{10}, \theta_9$

$P = 0, \mathrm{tr}\, Q = 0 \quad \longrightarrow \quad$ complete matrix invariant Q

$$\begin{aligned} \theta_4 &= -4 \det Q \\ \theta_{4.1} &= -32 (\det Q)'' \det Q - 36 ((\det Q)')^2 \\ \theta_6 &= 9 \det(Q') - 2 (\det Q)'' \\ \theta_{10} &= -4 \det Q \det(Q') + ((\det Q)')^2 \\ \theta_9 &= \det \begin{pmatrix} q_{11} & q_{12} & q_{21} \\ q'_{11} & q'_{12} & q'_{21} \\ q''_{11} & q''_{12} & q''_{21} \end{pmatrix} \end{aligned}$$

[2]

Osculating rank of the Plücker image

$$\alpha : \mathrm{Gr}_2(4) \to \mathrm{P}(\mathscr{N}) \subset \mathrm{P}(\mathbb{R}^{3,3}) \cong \mathrm{P}(\bigwedge^2 \mathbb{R}^4)$$

$$\mathscr{R}(t) \mapsto c(t)$$

$$D = \dim\langle c, c', \ldots, c^{(6)}\rangle \longleftarrow \text{ invariant}$$

$\mathscr{R}$ non-torsal $\Leftrightarrow c' \cdot c' \neq 0 \Rightarrow c \mapsto C \in \mathbb{R}^{3,3}$, $C' \cdot C' = 1$ (conformal lift)

relative conformal invariants:

$$\Delta_i = \det\left(\mathrm{Gram}\left(C, C', \ldots, C^{(i)}\right)\right)$$

$$\Delta_4 = \theta_4$$

Suggested classification

Test sample – 11 out of 13 classes of homogeneous non-torsal ruled surfaces (see [3])

- $D = \dim\langle c, c', \dots, c^{(6)}\rangle = 3$
  - quadric, doubly ruled
  - $Q = 0$, $\theta_i = 0$
- $D = 4, \Delta_4 = \theta_4 = 0$
  - $z = xy + h(x)$
  - $\det Q = 0$, $\theta_i = 0$
  - the flecnodal curves coincide
- $D = 4, \Delta_4 = \theta_4 \neq 0$
  - $z = g(x/y)$
  - Q diagonal, $\theta_9 = \theta_{10} = 0$
- $D = 5$
  - $\theta_9 = 0$
- $D = 6$
  - otherwise

Literature

[1] WILCZYNSKLI, Ernest J. , *Projective Differential Geometry of Curves and Ruled Surfaces*. Chelsea Publishing Company, S.D., 1906, p. 298.

[2] SASAKI, Takeshi, *Line congruence and transformation of projective surfaces*. Kyushu Journal of Mathematics, 2006, Volume 60.1: p. 101–243.

[3] DILLEN, Franki; SASAKI, Takeshi; VRANCKEN, Luc. *The classification of projectively homogeneous surfaces*. II. 1998.

[4] DOUBROV, Boris; ZELENKO, Igor, *Geometry of curves in generalized flag varieties*. Transformation Groups, 2013, Volume 18: p. 361–383.

[5] DOUBROV, Boris, *Generalized Wilczynski invariants for non-linear ordinary differential equations*. Springer, Symmetries and overdetermined systems of partial differential equations, 2008, p.25–40.