On full supergravity

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w/ Julian Kupka and Charles Strickland-Constable

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- Lorentzian metric $g_{\mu\nu}$ (graviton)
- 2-form $B_{\mu\nu}$ (Kalb–Ramond field)
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Action:

$$
\begin{aligned}
S = \int_M \sqrt{|g|} e^{-2\varphi} \Big(& R + 4|\nabla\varphi|^2 - \tfrac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \tfrac{1}{4} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}^\mu \not{\nabla} \psi_\mu + \bar{\rho} \not{\nabla} \rho \\
& + \tfrac{1}{2} \operatorname{Tr} \bar{\chi} \not{\nabla}_A \chi - 2 \bar{\psi}^\mu \nabla_\mu \rho + \tfrac{1}{4} \bar{\psi}^\mu \not{H} \psi_\mu - \tfrac{1}{4} \bar{\rho} \not{H} \rho - \tfrac{1}{8} \operatorname{Tr} \bar{\chi} \not{H} \chi \\
& + \tfrac{1}{2} H_{\mu\nu\rho} \bar{\psi}^\mu \gamma^\nu \psi^\rho + \tfrac{1}{4} \bar{\psi}^\mu H_{\mu\nu\rho} \gamma^{\nu\rho} \rho + \tfrac{1}{2} \operatorname{Tr} \bar{\chi} \not{F} \rho + \operatorname{Tr} F_{\mu\nu} \bar{\psi}^\mu \gamma^\nu \chi \\
& + \tfrac{1}{384} (\bar{\psi}_\mu \gamma_{\nu\rho\sigma} \psi^\mu)(\bar{\rho} \gamma^{\nu\rho\sigma} \rho) - \tfrac{1}{768} (\bar{\rho} \gamma^{\mu\nu\rho} \rho) \operatorname{Tr}(\bar{\chi} \gamma_{\mu\nu\rho} \chi) \\
& - \tfrac{1}{192} (\bar{\psi}_\mu \gamma_{\rho\sigma\tau} \psi^\mu)(\bar{\psi}_\nu \gamma^{\rho\sigma\tau} \psi^\nu) + \tfrac{1}{192} (\bar{\psi}_\mu \gamma_{\nu\rho\sigma} \psi^\mu) \operatorname{Tr}(\bar{\chi} \gamma^{\nu\rho\sigma} \chi) \\
& - \tfrac{1}{768} \operatorname{Tr}(\bar{\chi} \gamma_{\mu\nu\rho} \chi) \operatorname{Tr}(\bar{\chi} \gamma^{\mu\nu\rho} \chi) \Big)
\end{aligned}
$$

Supersymmetry variations

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$$\delta g_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}$$

$$\delta B_{\mu\nu} = \bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} - \operatorname{Tr} A_{[\mu}\bar{\epsilon}\gamma_{\nu]}\chi$$

$$\delta A_{\mu} = -\tfrac{1}{2}\bar{\epsilon}\gamma_{\mu}\chi$$

$$\delta\varphi = \tfrac{1}{4}\bar{\rho}\epsilon - \tfrac{1}{4}\bar{\psi}^{\mu}\gamma_{\mu}\epsilon$$

$$\delta\rho = -\nabla\!\!\!/\,\epsilon + (\nabla_{\mu}\varphi)\gamma^{\mu}\epsilon + \tfrac{1}{4}H\!\!\!/\,\epsilon + \tfrac{1}{96}(\bar{\psi}_{\mu}\gamma_{\nu\rho\sigma}\psi^{\mu})\gamma^{\nu\rho\sigma}\epsilon + \tfrac{1}{4}(\bar{\rho}\epsilon)\rho - \tfrac{1}{192}\operatorname{Tr}(\bar{\chi}\gamma_{\mu\nu\rho}\chi)\gamma^{\mu\nu\rho}\epsilon$$

$$\delta\psi_{\mu} = \nabla_{\mu}\epsilon - \tfrac{1}{8}H_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon - \tfrac{1}{4}(\bar{\psi}_{\mu}\rho)\epsilon - \tfrac{1}{4}(\bar{\psi}_{\mu}\gamma_{\nu}\epsilon)\gamma^{\nu}\rho + \tfrac{1}{4}(\bar{\rho}\epsilon)\psi_{\mu}$$

$$\delta\chi = \tfrac{1}{2}F\!\!\!/\,\epsilon - \tfrac{1}{4}(\bar{\chi}\rho)\epsilon - \tfrac{1}{4}(\bar{\chi}\gamma_{\mu}\epsilon)\gamma^{\mu}\rho + \tfrac{1}{4}(\bar{\rho}\epsilon)\chi,$$

[Bergshoeff–de Roo–de Wit–van Niewenhuizen '82] [Chapline–Manton '83] [Dine–Rohm–Seiberg–Witten '85]

Supergravity in generalised geometry

[Kupka–Strickland-Constable–V '24]

Supergravity in generalised geometry

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$$S = \int_M \mathcal{R}\sigma^2 + \bar{\psi}_\alpha \not{D}\psi^\alpha + \bar{\rho}\not{D}\rho + 2\bar{\rho}D_\alpha\psi^\alpha - \tfrac{1}{768}\sigma^{-2}(\bar{\psi}_\alpha\gamma_{cde}\psi^\alpha)(\bar{\rho}\gamma^{cde}\rho) - \tfrac{1}{384}\sigma^{-2}(\bar{\psi}_\alpha\gamma_{cde}\psi^\alpha)(\bar{\psi}_\beta\gamma^{cde}\psi^\beta)$$

$$\begin{aligned}
\delta\mathcal{G}_{ab} = \delta\mathcal{G}_{\alpha\beta} = 0, &\quad \delta\mathcal{G}_{a\beta} = \delta\mathcal{G}_{\beta a} = \tfrac{1}{2}\sigma^{-2}\bar{\epsilon}\gamma_a\psi_\beta \\
\delta\sigma &= \tfrac{1}{8}\sigma^{-1}(\bar{\rho}\epsilon) \\
\delta\rho &= \not{D}\epsilon + \tfrac{1}{192}\sigma^{-2}(\bar{\psi}_\alpha\gamma_{cde}\psi^\alpha)\gamma^{cde}\epsilon \\
\delta\psi_\alpha &= D_\alpha\epsilon + \tfrac{1}{8}\sigma^{-2}(\bar{\psi}_\alpha\rho)\epsilon + \tfrac{1}{8}\sigma^{-2}(\bar{\psi}_\alpha\gamma_c\epsilon)\gamma^c\rho
\end{aligned}$$

Generalised geometry crashcourse

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Transitive Courant algebroid (local picture) [Ševera '90s]:

- Data: manifold $M = \mathbb{R}^{10}$, Lie algebra $\mathfrak{g}$ with an invariant pairing Tr
- Vector bundle: $E = TM \oplus T^*M \oplus (\mathfrak{g} \times M)$
- Fiberwise inner product $\langle x + \alpha + s, y + \beta + t \rangle := \alpha(y) + \beta(x) + \mathrm{Tr}(st)$
- Bracket: $[x + \alpha + s, y + \beta + t] = L_x y + (L_x \beta - i_y d\alpha + \mathrm{Tr}\, t\, ds) + (L_x t - L_y s + [s, t]_{\mathfrak{g}})$
- Vector bundle map (anchor): $a\colon E \to TM$, $a(x + \alpha + s) = x$

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Bosonic fields [Coimbra–Strickland-Constable–Waldram '11]:

- Generalised metric = symmetric endomorphism $\mathcal{G}\colon E \to E$ s.t. $\mathcal{G}^2 = 1$

  $\rightsquigarrow$ orthogonal splitting $E = C_+ \oplus C_-$ (denote the frames by e_a and e_α)

  $\rightsquigarrow$ encodes g, B, A via $C_+ = \mathrm{graph}(g + B + A\colon T \to T^* \oplus \mathfrak{g})$

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- Nonzero half-density $\sigma \in \Gamma(H)$
  $\rightsquigarrow$ encodes the dilaton φ via $\sigma^2 = \sqrt{|g|}e^{-2\varphi}$

Levi-Civita connections:

- connection $D\colon \Gamma(E)\times\Gamma(E)\to\Gamma(E)$ s.t.
  $D_{fu}v = fD_u v,\quad D_u(fv) = fD_u v + (a(u)f)v,\;\; D\langle\,\cdot\,,\,\cdot\,\rangle = 0$
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- denote the Majorana–Weyl spinor bundles for C_+ by $S_\pm$
- generalised dilatino $\rho\in\Gamma(\Pi S_+\otimes H)$ $\rightsquigarrow$ corresponds to ρ
- generalised gravitino $\psi\in\Gamma(\Pi S_-\otimes C_-\otimes H)$ $\rightsquigarrow$ corresponds to ψ and χ
- supersymmetry parameter $\epsilon\in\Gamma(\Pi S_-\otimes H)$

Supergravity in generalised geometry again

[Kupka–Strickland-Constable–V '24]

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$$(\bar\lambda_1\psi_1)(\bar\lambda_2\psi_2) = \tfrac{1}{16}(\bar\lambda_1\gamma_{(1)}\lambda_2)(\bar\psi_1\gamma^{(1)}\psi_2) + \tfrac{1}{96}(\bar\lambda_1\gamma_{(3)}\lambda_2)(\bar\psi_1\gamma^{(3)}\psi_2) + \tfrac{1}{3840}(\bar\lambda_1\gamma_{(5)}\lambda_2)(\bar\psi_1\gamma^{(5)}\psi_2)$$

$$(\bar\lambda_1\gamma_a\lambda_2)(\bar\lambda_3\gamma^a\lambda_4) = \tfrac{1}{2}(\bar\lambda_1\gamma_a\lambda_3)(\bar\lambda_2\gamma^a\lambda_4) + \tfrac{1}{24}(\bar\lambda_1\gamma_{(3)}\lambda_3)(\bar\lambda_2\gamma^{(3)}\lambda_4)$$

$$(\bar\lambda_1\gamma_a\lambda_2)(\bar\psi_1\gamma^a\psi_2) = \tfrac{5}{8}(\bar\lambda_1\psi_1)(\bar\lambda_2\psi_2) + \tfrac{3}{16}(\bar\lambda_1\gamma_{(2)}\psi_1)(\bar\lambda_2\gamma^{(2)}\psi_2) + \tfrac{1}{192}(\bar\lambda_1\gamma_{(4)}\psi_1)(\bar\lambda_2\gamma^{(4)}\psi_2)$$

$$(\bar\lambda\gamma^{ab}\psi_1)(\bar\psi_2\gamma_a\psi_3) = -\tfrac{7}{16}(\bar\lambda\gamma^{bc}\psi_2)(\bar\psi_1\gamma_c\psi_3) - \tfrac{9}{16}(\bar\lambda\psi_2)(\bar\psi_1\gamma^b\psi_3) - \tfrac{1}{32}(\bar\lambda\gamma^{bcde}\psi_2)(\bar\psi_1\gamma_{cde}\psi_3) - \tfrac{5}{32}(\bar\lambda\gamma_{cd}\psi_2)(\bar\psi_1\gamma^{bcd}\psi_3) - \tfrac{1}{384}(\bar\lambda\gamma_{cdef}\psi_2)(\bar\psi_1\gamma^{bcdef}\psi_3)$$

$$\tfrac{1}{2}(\bar\lambda\gamma^{d[ab}\lambda)\bar\lambda\gamma_d\gamma^{c]} = (\bar\lambda\gamma^{abc}\lambda)\bar\lambda \qquad (\bar\lambda\gamma^{abc}\lambda)\bar\lambda\gamma_{ab} = 0 \qquad (\bar\lambda\gamma^{abc}\lambda)\bar\lambda\gamma_{abc} = 0$$

Conclusions and remarks

- Completion of the program of writing supergravity via generalised geometry.
- Allows the check of supersymmetry by *hand*.
- Simplification even when returning back to ordinary variables (only γ_{abc}).
- Confirmation that gen. geometry provides natural variables for string massless sector.

- Generalised Lichnerowitz identity: $\not{D}^2 + D^\alpha D_\alpha = -\frac{1}{8}\mathcal{R}$ on spinor half-densities

- Case $\mathcal{G} = 1 \rightsquigarrow$ dilatonic gravity (topological locally supersymmetric theory for σ & ρ)
- Case $M = \{*\} \rightsquigarrow$ finite-dimensional toy model of supergravity