The crossed product calculus

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Motivation and references

Goal of the talk: explore **noncommutative differential geometry** (as introduced by Antonio) of a particular class of **quantum principal bundles**.

$$
\begin{array}{ccccc}
\begin{pmatrix}\text{trivial}\\ \text{extensions}\end{pmatrix} & \subsetneq & \begin{pmatrix}\text{cleft}\\ \text{extensions}\end{pmatrix} & \subsetneq & \begin{pmatrix}\text{Hopf–Galois}\\ \text{extensions}\end{pmatrix}\\
\Updownarrow & & \Updownarrow & \leftarrow & \text{[Doi–Takeuchi]}\\
\begin{pmatrix}\text{smash product}\\ \text{algebras}\end{pmatrix} & \subsetneq & \begin{pmatrix}\text{crossed product}\\ \text{algebras}\end{pmatrix} & &
\end{array}
$$

Noncommutative differential geometry for

- **smash product algebras**: Pflaum, M., Schauenburg, P.: *Differential calculi on noncommutative bundles.* Z Phys C-Particles and Fields **76** (1997) 733–744.
- crossed product algebras: Sciandra, A., TW: *Noncommutative differential geometry on crossed product algebras.* J. Algebra. **664** (2025) 129–176.

Very rough idea

Quantum Principal Bundle

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A \otimes H \\ \uparrow & & \\ B & & \end{array}$$

Let's assume that A **trivializes**: $A \cong B \otimes H$ as right H-comodule algebra.

Then, given differential calculi $\Omega^\bullet(B)$, $\Omega^\bullet(H)$ we define a differential calculus on A via

$$\Omega^n(A) = \bigoplus_{k=0}^{n} \Omega^{n-k}(B) \otimes \Omega^k(H)$$

with differential $\mathrm{d}_\otimes(\omega_B \otimes \omega) := \mathrm{d}_B(\omega_B) \otimes \omega + (-1)^{|\omega_B|}\omega_B \otimes \mathrm{d}_H\omega$ and wedge product $(\omega_B \otimes \omega) \wedge (\eta_B \otimes \eta) := (-1)^{|\omega||\eta_B|}\omega_B \wedge \eta_B \otimes \omega \wedge \eta$.

- smash product algebra: $B\#H$ for action $H \otimes B \to B$
- crossed product algebra: $B\#_\sigma H$ for weak action and 2-cocycle $\sigma\colon H \otimes H \to B$

H Hopf algebra with coproduct $\Delta\colon H \to H \otimes H$, counit $\varepsilon\colon H \to \mathbb{k}$ and antipode $S\colon H \to H$.

Sweedler's notation:

$$\begin{aligned} \Delta(h) &=: h_1 \otimes h_2, \\ (\Delta \otimes \mathrm{id})(\Delta(h)) = (\mathrm{id} \otimes \Delta)(\Delta(h)) &=: h_1 \otimes h_2 \otimes h_3, \\ &\vdots \end{aligned}$$

B (associative unital) algebra with H-action $\cdot\colon H \otimes B \to B$ such that

$$h \cdot (bb') = (h_1 \cdot b)(h_2 \cdot b'), \qquad h \cdot 1_B = \varepsilon(h) 1_B$$

$\rightsquigarrow$ B is called a **left H-module algebra**.

Lemma

$B \otimes H$ is an (associative unital) algebra w.r.t. the smash product

$$(b \otimes h)(b' \otimes h') := b(h_1 \cdot b') \otimes h_2 h'.$$

*We call it the **smash product algebra** and write $B\#H$.*

Trivial extensions

Let A be a right H-comodule algebra, i.e. $\Delta_A \colon A \to A \otimes H$ is algebra morphism.
$\rightsquigarrow B := A^{\mathrm{co}H} := \{b \in A \mid \Delta_A(b) = b \otimes 1\} \subseteq A$ is a subalgebra.

We say that $B \subseteq A$ is a **trivial extension** if $\exists$ convolution invertible comodule algebra morphism $j \colon H \to A$, the **cleaving map**.

In particular, $\exists\, j^{-1} \colon H \to A$ s.t. $j(h_1)j^{-1}(h_2) = \varepsilon(h)1_A = j^{-1}(h_1)j(h_2)\ \forall h \in H$.

Proposition (Doi–Takeuchi '86, Part I)

Let A be a right H-comodule algebra. Then A is (isomorphic in $\mathcal{A}^H$ to) a smash product algebra if and only if $B := A^{\mathrm{co}H} \subseteq A$ is a trivial extension.

Proof.

If $A \cong B\#H \Rightarrow j \colon H \to B\#H$, $j(h) := 1\#h$ is a cleaving map.

If $j \colon H \to A$ is cleaving map $\Rightarrow h \cdot b := j(h_1)bj^{-1}(h_2)$ is left H-module algebra structure and

$$A \to B\#H, \qquad a \mapsto a_0 j^{-1}(a_1)\#a_2$$

is isomorphism in $\mathcal{A}^H$ with inverse $b\#h \mapsto b \cdot h$. □

(Co)action-compatible differential calculi

- A FODC $(\Omega^1(H), \mathrm{d}_H)$ on H is called **left H-covariant** if $\Omega^1(H) \in {}_H\mathcal{M}_H$ is left H-covariant and $\mathrm{d}_H \colon H \to \Omega^1(H)$ is left H-colinear, i.e.

$$\lambda_{\Omega^1}(h\omega h') = \Delta(h)\lambda_{\Omega^1}(\omega)\Delta(h'), \qquad \lambda_{\Omega^1} \circ \mathrm{d}_H = (\mathrm{id} \otimes \mathrm{d}_H) \circ \Delta,$$

  where $\lambda_{\Omega^1} : \Omega^1(H) \to H \otimes \Omega^1(H)$ is left coaction, $h, h' \in H$, $\omega \in \Omega^1(H)$.
- A FODC $(\Omega^1(B), \mathrm{d}_B)$ on a left H-module algebra B is called **left H-equivariant** if $\Omega^1(B) \in {}_B\mathcal{M}_B$ is left H-equivariant and $\mathrm{d}_B \colon B \to \Omega^1(B)$ is left H-linear, i.e.

$$h \cdot (b\omega_B b') = (h_1 \cdot b)(h_2 \cdot \omega_B)(h_3 \cdot b'), \qquad \mathrm{d}_B(h \cdot b) = b \cdot \mathrm{d}_B(b)$$

  for all $b, b' \in B$, $h \in H$ and $\omega_B \in \Omega^1(B)$.

Theorem (Pflaum–Schauenburg '97)

Given $(\Omega^1(H), \mathrm{d}_H)$ and $(\Omega^1(B), \mathrm{d}_B)$ as above we define a FODC $(\Omega^1(B\#H), \mathrm{d}_\#)$ on $B\#H$ by

$$\Omega^1(B\#H) := (\Omega^1(B) \otimes H) \oplus (B \otimes \Omega^1(H))$$

with $\mathrm{d}_\# := \mathrm{d}_B + \mathrm{d}_H$ and $B\#H$-module actions

$$(b\#h)(\omega_B \otimes h' + b' \otimes \omega) := b(h_1 \cdot \omega_B) \otimes h_2 h' + b(h_1 \cdot b') \otimes h_2\omega,$$
$$(\omega_B \otimes h + b \otimes \omega)(b' \otimes h') := \omega_B(h_1 \cdot b') \otimes h_2 h' + b(\omega_{-1} \cdot b') \otimes \omega_0 h'.$$

We call $(\Omega^1(B\#H), \mathrm{d}_\#)$ the **smash product calculus**.

Corollary

If $(\Omega^1(H), \mathrm{d}_H)$ is bicovariant (instead of only left covariant) then the FODC $(\Omega^1(B\#H), \mathrm{d}_\#)$ is right H-covariant w.r.t.

$$\Delta_{\Omega^1(B\#H)} := \mathrm{id} \otimes (\Delta + \Delta_{\Omega^1(H)}) \colon \Omega^1(B\#H) \to \Omega^1(B\#H) \otimes H.$$

Considering (higher order) differential calculi $\Omega^\bullet(H), \Omega^\bullet(B)$ s.t. $\Omega^\bullet(H)$ is left covariant and $\Omega^\bullet(B)$ left H-equivariant we obtain

Theorem (Pflaum–Schauenburg '97)

$\Omega^\bullet(B\#H) := \bigoplus_{n\geq 0} \Omega^n(B\#H)$ *with* $\Omega^n(B\#H) := \bigoplus_{k=0}^n \Omega^{n-k}(B) \otimes \Omega^k(H)$ *is a differential calculus on* $B\#H$ *with* $\mathrm{d}_\#(\omega_B \otimes \omega) := \mathrm{d}_B\omega_B \otimes \omega + (-1)^{|\omega_B|}\omega_B \otimes \mathrm{d}_H\omega$,

$$(\omega_B \otimes \omega) \wedge (\eta_B \otimes \eta) := (-1)^{|\omega||\eta_B|}\omega_B \wedge (\omega_{-1} \cdot \eta_B) \otimes \omega_0 \wedge \eta.$$

Proposition (DelDonno-Latini-TW '24)

*If $\Omega^\bullet(H)$ is the maximal prolongation of a bicovariant FODC then $\Omega^\bullet(B\#H)$ is **complete** in the sense of Đurđević, i.e. $\Delta_{B\#H}$ extends to a DGA morphism*

$$\Delta^\bullet_{B\#H} \colon \Omega^\bullet(B\#H) \to \Omega^\bullet(B\#H) \otimes \Omega^\bullet(H).$$

Example

Let $(H, \mathcal{R})$ be a **coquasitriangular Hopf algebra** with universal $\mathcal{R}$-form $\mathcal{R}\colon H \otimes H \to \mathbb{k}$. Then every left H-comodule B becomes a left H-module via

$$(h \cdot b) := \mathcal{R}(b_{-1} \otimes h) b_0.$$

If B is a left H-comodule algebra then it becomes a left H-module algebra in this way.

- $B \# H$ reads $(b\#h)(b'\#h') = b(h_1 \cdot b')\#h_2h' = b\mathcal{R}(b'_{-1} \otimes h_1)b'_0\#h_2h'$.
- for left H-covariant $\Omega^\bullet(H), \Omega^\bullet(B)$ we obtain $\Omega^\bullet(B\#H)$ with

$$(\omega_B \otimes \omega) \wedge (\eta_B \otimes \eta) = (-1)^{|\omega||\eta_B|}\mathcal{R}((\eta_B)_{-1} \otimes \omega_{-1})\omega_B \wedge (\eta_B)_0 \otimes \omega_0 \wedge \eta.$$

For instance, take a braided vector space (V, τ) and the coquasitriangular bialgebra H_τ constructed by **[Faddeev-Reshetikhin-Takhtajan '90]**. Take a quotient Hopf algebra $H_\sigma \to H$ (the matrix quantum groups arise this way!).

H is generated by T^i_j modulo $R^{ji}_{k\ell} T^k_m T^\ell_n = T^i_k T^j_\ell R^{\ell k}_{mn}$.
Define B as the free algebra generated by x_i with left H-coaction $x_i \mapsto T^j_i \otimes x_j$.

Then $\Omega^\bullet(B\#H)$ coincides with the calculi studied by [Wess-Zumino '91].

2-cocycles and twisted actions

Twisted version of smash product with a 2-**cocycle** $\sigma\colon H\otimes H\to B$

i.) σ is **convolution invertible**, i.e. $\exists\ \sigma^{-1}\colon H\otimes H\to B$ s.t. for all $h,h'\in H$

$$\sigma(h_1\otimes h'_1)\sigma^{-1}(h_2\otimes h'_2)=\varepsilon(hh')1_B=\sigma^{-1}(h_1\otimes h'_1)\sigma(h_2\otimes h'_2)$$

ii.) σ is **normalized**, i.e. $\sigma(1\otimes h)=\varepsilon(h)1_B=\sigma(h\otimes 1)$

iii.) σ satisfies the 2-**cocycle property**

$$(h_1\cdot\sigma(h'_1\otimes h''_1))\sigma(h_2\otimes h'_2h''_2)=\sigma(h_1\otimes h'_1)\sigma(h_2h'_2\otimes h'')$$

A σ-**twisted left** H-**action** is a $\mathbb{k}$-linear map $\cdot\colon H\otimes B\to B$ s.t. $1\cdot b=b$ and

$$h\cdot(h'\cdot b)=\sigma(h_1\otimes h'_1)((h_2h'_2)\cdot b)\sigma^{-1}(h_3\otimes h'_3)$$

for all $h,h'\in H$ and $b\in B$.

Definition

Let $\sigma\colon H\otimes H\to B$ be a 2-cocycle. Then B is called a σ-**twisted left** H-**module algebra** if there is a σ-twisted left H-action $\cdot\colon H\otimes B\to B$ s.t. $h\cdot 1_B=\varepsilon(h)1_B$ and

$$h\cdot(bb')=(h_1\cdot b)(h_2\cdot b').$$

The crossed product algebra

Proposition

For a σ-twisted left H-module algebra B we define an associative product on $B \otimes H$ by

$$(b \otimes h)(b' \otimes h') := b(h_1 \cdot b')\sigma(h_2 \otimes h'_1) \otimes h_3 h'_2$$

*with unit $1_B \otimes 1$. This is the **crossed product algebra** and we write $B\#_\sigma H$.*

Remark

We can always define the trivial 2-cocycle $\sigma_\varepsilon(h \otimes h') := \varepsilon(hh')1_B$.
If B is a left H-module algebra then it is a σ_ε-twisted left H-module algebra and

$$B\#_{\sigma_\varepsilon} H = B\#H$$

is the smash product algebra.

Theorem (Doi–Takeuchi '86, Part II)

Let A be a right H-comodule algebra. Then A is (isomorphic in $\mathcal{A}^H$ to) a crossed product algebra if and only if $B := A^{\mathrm{co}H} \subseteq A$ is a cleft extension.

Crossed product calculus

Fix a σ-twisted left H-module algebra B.

Definition

A differential calculus $\Omega^\bullet(B)$ is called **σ-twisted** if for all $n > 0$ there are $\Bbbk$-linear maps $\cdot\colon H \otimes \Omega^n(B) \to \Omega^n(B)$ s.t. for all $b^0, \ldots, b^n \in B$

i.) $h \cdot (b^0 \mathrm{d}(b^1) \wedge \ldots \wedge \mathrm{d}(b^n)) = (h_1 \cdot b^0)\mathrm{d}(h_2 \cdot b^1) \wedge \ldots \wedge \mathrm{d}(h_{n+1} \cdot b^n)$

ii.) $\mathrm{d} \circ \sigma = 0$

Lemma

$$h \cdot (h' \cdot \omega_B) = \sigma(h_1 \otimes h'_1)((h_2 h'_2) \cdot \omega_B)\sigma^{-1}(h_3 \otimes h'_3)$$

Theorem (Sciandra-TW '25)

For $\Omega^\bullet(H)$ left covariant and $\Omega^\bullet(B)$ σ-twisted we have a differential calculus $\Omega^n(B \#_\sigma H) := \bigoplus_{k=0}^n \Omega^{n-k}(B) \otimes \Omega^k(H)$ with $\mathrm{d}_{\#_\sigma} = \mathrm{d}_B + \mathrm{d}_H$ and

$$(\omega_B \otimes \omega) \wedge (\eta_B \otimes \eta) := (-1)^{|\omega||\eta_B|}\omega_B(\omega_{-2} \cdot \eta_B)\sigma(\omega_{-1} \otimes \eta_{-1}) \otimes \omega_0\eta_0$$

Explicit examples on **pointed Hopf algebras** and the **noncommutative** 2-**torus** can be found in [Sciandra-TW '25].

Theorem (DelDonno-Latini-TW '24)

*If $\Omega^\bullet(H)$ is max. prolongation of **bi**covariant FODC and $\Omega^\bullet(B)$ σ-twisted then $\Omega^\bullet(B\#_\sigma H)$ is **complete** in the sense of Đurđević. Moreover,*

- *the base forms coincide with $\Omega^\bullet(B)$.*
- *vertical and horizontal forms are $\mathrm{ver}^\bullet = B \otimes \Omega^\bullet(H)$ and $\mathrm{hor}^\bullet = \Omega^\bullet(B) \otimes H$.*
- *the Đurđević braiding reads*

$$\sigma^{\mathrm{Dur}}((b \otimes h) \otimes_B (b' \otimes h')) \\ = (b(h_1 \cdot b')\sigma(h_2 \otimes h'_1)(h_3 h'_2 \cdot \sigma^{-1}(S(h_8) \otimes h_9))\sigma(h_4 h'_3 \otimes S(h_7)) \otimes h_5 h'_4 S(h_6)) \otimes_B (1 \otimes h_{10})$$

In particular, the Atiyah sequence is exact in the category ${}_{B\#_\sigma H}\mathcal{M}^H_{B\#_\sigma H}$.

$$0 \to \Omega^1(B) \otimes H \to \Omega^1(B\#_\sigma H) \to B \otimes \Omega^1(H) \to 0$$

Proposition (Sciandra-TW '25)

The crossed product calculus admits a strong connection

$$c\colon (B\#_\sigma H) \otimes {}^{\mathrm{co}H}\Omega^1(H) \to \Omega^1(B\#_\sigma H), \qquad c((b \otimes h) \otimes \omega) := b \otimes h\omega.$$

The induced covariant derivative on associated bundles (for $V \in \mathcal{M}^H$) reads

$$\nabla_E\colon E := ((B\#_\sigma H) \otimes V)^{\mathrm{co}H} \to \Omega^1(B) \otimes_B E, \qquad \nabla_E(b \otimes h \otimes v) := \mathrm{d}_B b \otimes h \otimes v.$$

References

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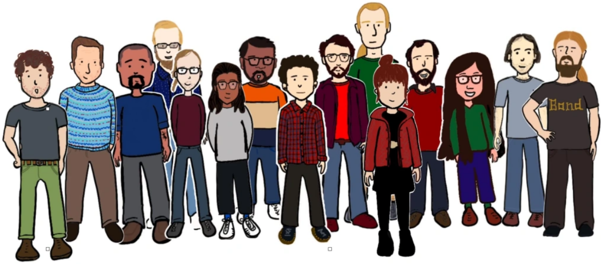

Figure: The group drawn by Karen

Thank you for your attention!