Higher order connections in noncommutative geometry

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Differentiable algebras

Let A be a unital associative algebra over a commutative ring k.

Definition

An exterior algebra, or differential calculus, for A, is a differential graded algebra

$$(\Omega_d^\bullet, d, \wedge)$$

where d has degree 1, satisfying the surjectivity condition

$$\Omega_d^0 = A, \quad \Omega_d^\bullet = \langle dA \rangle$$

Let's call (A, d) a differentiable algebra.
Our notions of noncommutative differential geometry will take place in the category ${}_A\mathrm{Mod}$ of left modules over A.

Jet functors in noncommutative geometry

A differentiable algebra gives rise to a family of endofunctors J^n_d on ${}_A\mathrm{Mod}$, which we call the jet functors. Natural transformations ($n > m > h$):

$$j^n_d\colon \mathrm{id} \longrightarrow J^n_d \qquad \pi^{n,m}_d\colon J^n_d \longrightarrow J^m_d,$$

such that

$$\pi^{n,m}_d \circ \pi^{m,h}_d = \pi^{n,h}_d, \qquad \pi^{n,m}_d \circ j^n_d = j^m_d.$$

Note that we have $J^0_d = \mathrm{id}$.

For more details on this construction see the poster by M.Mantegazza!

Symmetric forms and jet functors

We define the functors

$$S_d^0 = \Omega_d^0 = \mathrm{id}, \qquad S_d^1 = \Omega_d^1 := \Omega_d^1 \otimes_A -.$$

For $n \geq 0$, the functor of symmetric forms S_d^n is defined by induction as the kernel of the following composition

$$\Omega_d^1 \circ S_d^{n-1} \xrightarrow{\Omega_d^1(\iota_\wedge^{n-1})} \Omega_d^1 \circ \Omega_d^1 \circ S_d^{n-2} \xrightarrow{\wedge_{S_d^{n-2}}} \Omega_d^2 \circ S_d^{n-2}$$

and $\iota_\wedge^n \colon S_d^n \longrightarrow \Omega_d^1 \circ S_d^{n-1}$ is the inclusion.

- Natural transformation:

$$\iota_d^n \colon S_d^n \longrightarrow J_d^n \qquad \pi_d^{n,m} \circ \iota_d^n = 0,$$

Classical correspondence

Theorem

Let $A = C^\infty(M)$ and d be the usual exterior derivative. Let E be a vector bundle over M. Then $J^n_d(\Gamma(E)) = \Gamma(J^n(M, E))$, and all the natural transformations coincide with their classical version.

Remark

One of our guiding principles is a semantic correspondence, in that our NCG propositions and formulas remain valid when interpreted in the obvious differential geometric way.

Differential operators

Definition

Let $E, F \in {}_A\mathrm{Mod}$. A k-linear map $\Delta\colon E \to F$ is called a linear differential operator of order at most n with respect to the exterior algebra $\Omega_d^\bullet$, if there exists an $\widetilde{\Delta} \in {}_A\mathrm{Hom}(J_d^n E, F)$ such that the following diagram commutes:

$$\begin{array}{ccc} J_d^n E & & \\ {\scriptstyle j_{d,E}^n}\uparrow & \searrow^{\widetilde{\Delta}} & \\ E & \xrightarrow{\Delta} & F \end{array}$$

If n is minimal, we say that Δ is a linear differential operator of order n.

Algebras and category of differential operators

Proposition

Let $n \leq m$, then a differential operator of order at most n is also a differential operator of order at most m.

Proposition

Let $\Delta_1 \colon E \to F$ and $\Delta_2 \colon F \to G$ be differential operators of order at most n and m, respectively. Then the composition $\Delta_2 \circ \Delta_1 \colon E \to G$ is a differential operator of order at most $n + m$.

Corollary

The differential operators form a category with objects left A-modules, and $\mathrm{Diff}_d(E, E)$ is a unital associative k-algebra for each E.

Definition (Symbol of a differential operator)

Let E and F be in ${}_A\mathrm{Mod}$, we define

$$\mathrm{Symb}^n_d(E,F) := \mathrm{Diff}^n_d(E,F)/\,\mathrm{Diff}^{n-1}_d(E,F),$$

for all $n \geq 0$, with the convention $\mathrm{Diff}^{-1}_d(E,F) = 0$. We call the quotient projection $\varsigma^n_d\colon\ \mathrm{Diff}^n_d(E,F) \to \mathrm{Symb}^n_d(E,F)$ the symbol map, and the symbol of $\Delta \in \mathrm{Diff}^n_d(E,F)$ is the equivalence class $\varsigma^n_d(\Delta)$ containing Δ.

If $\mathrm{im}(\iota^n_{d,E}) \subseteq Aj^n_{d,E}(E)$, then the mapping $r^n_{d,E,F}$ defined by

$$r^n_{d,E,F}\colon\ \mathrm{Symb}^n_d(E,F) \longrightarrow {}_A\mathrm{Hom}(S^n_d(E),F), \qquad \varsigma^n_d(\Delta) \longmapsto \widetilde{\Delta} \circ \iota^n_{d,E}.$$

is well-defined and natural in E and F.

Symbol algebras

Proposition

For each A-module E, $\mathrm{Symb}_d(E, E)$ *inherits a graded product from* $\mathrm{Diff}_d(E, E)$*, and is a unital associative k-algebra.*

Remark

Classically, $\mathrm{Symb}_d(C^\infty(M), C^\infty(M)) = C^\infty_{poly}(T^*M)$*, functions on the total space of the cotangent bundle which are polynomial in fibres.*

In the noncommutative setting, these symbol algebras are not commutative algebras, but they are "symmetric" in the same sense as the symmetric forms.

$$\forall a \in A, \quad \sum_{i,j} \partial_i \circ \partial_j(a)\, \theta_i \otimes_A \theta_j \in S^2_d$$

whenever Ω^1_d is parallelizable.

Spencer operators

Definition

The Spencer operators are natural transformations for $n \geq 1$ and $m \geq 0$ with component at E in ${}_A\mathrm{Mod}$

$$\mathcal{S}^{n,m}_{d,E}\colon \Omega^m_d J^n_d E \longrightarrow \Omega^{m+1}_d J^{n-1}_d E,$$
$$\omega \otimes_A \sum_j y_j j^1_d(z_j) \otimes_A \xi_j \longmapsto \sum_j d(\omega y_j) z_j \otimes_A \xi_j,$$

for all $\omega \in \Omega^m_d$ and $\sum_j y_j j^1_d(z_j) \otimes_A \xi_j \in J^n_d E \subseteq J^1_d J^{n-1}_d E$.

Proposition (Classical Spencer)

The Spencer operator coincides with the operator given by the formula

$$\omega \otimes_A \xi \longmapsto d\omega \otimes_A \pi^{n,n-1}_{d,E}(\xi) + (-1)^{\deg(\omega)} \omega \wedge \mathcal{S}^{n,0}_{d,E}(\xi).$$

Spencer bicomplex

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{id}_{A\mathsf{Mod}} & = & \mathrm{id}_{A\mathsf{Mod}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow j^{n}_{d} & & \downarrow j^{n-1}_{d} & & \\
0 & \longrightarrow & S^{n}_{d} & \xrightarrow{\iota^{n}_{d}} & J^{n}_{d} & \xrightarrow{\pi^{n,n-1}_{d}} & J^{n-1}_{d} & \longrightarrow & 0 \\
 & & -\delta^{n,0}_{d} \downarrow & & \downarrow \mathcal{S}^{n,0}_{d} & & \downarrow \mathcal{S}^{n-1,0}_{d} & & \\
0 & \to & \Omega^{1}_{d} S^{n-1}_{d} & \xrightarrow{\Omega^{1}_{d}(\iota^{n-1}_{d})} & \Omega^{1}_{d} J^{n-1}_{d} & \xrightarrow{\Omega^{1}_{d}(\pi^{n-1,n-2}_{d})} & \Omega^{1}_{d} J^{n-2}_{d} & \to & 0 \\
 & & -\delta^{n-1,1}_{d} \downarrow & & \downarrow \mathcal{S}^{n-1,1}_{d} & & \downarrow \mathcal{S}^{n-2,1}_{d} & & \\
0 & \to & \Omega^{2}_{d} S^{n-2}_{d} & \xrightarrow{\Omega^{2}_{d}(\iota^{n-2}_{d})} & \Omega^{2}_{d} J^{n-2}_{d} & \xrightarrow{\Omega^{2}_{d}(\pi^{n-2,n-3}_{d})} & \Omega^{2}_{d} J^{n-3}_{d} & \to & 0 \\
 & & -\delta^{n-2,2}_{d} \downarrow & & \downarrow \mathcal{S}^{n-2,2}_{d} & & \downarrow \mathcal{S}^{n-3,2}_{d} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Jet exact sequence

Theorem (Holonomic jet exact sequence)

Let A be a k-algebra endowed with an exterior algebra $\Omega_d^\bullet$ such that Ω_d^1, Ω_d^2, and Ω_d^3 are flat in Mod_A. For $n \geq 1$, if the Spencer δ cohomology $H^{m,2}$ vanishes, for all $1 \leq m < n-2$, then the following sequence is exact,

$$0 \longrightarrow S_d^n \xhookrightarrow{\iota_d^n} J_d^n \xrightarrow{\pi_d^{n,n-1}} J_d^{n-1} \longrightarrow H^{n-2,2}.$$

Therefore, if $H^{n-2,2} = 0$ we obtain a short exact sequence

$$0 \longrightarrow S_d^n \xhookrightarrow{\iota_d^n} J_d^n \xrightarrow{\pi_d^{n,n-1}} J_d^{n-1} \longrightarrow 0.$$

From now on, we will assume that the jet sequence is exact for all n, and also that $J_d^n A = A j_d^n(A)$.

Connections on modules

Definition

A left connection on E is a k-linear map $\nabla \colon E \to \Omega^1_d E$ satisfying

$$\nabla(ae) = da \otimes_A e + a\nabla(e)$$

Proposition

A map ∇ is a left connection if and only if it is a differential operator of order 1 with restriction symbol $\mathrm{id}_{\Omega^1_d E}$, i.e.

$$0 \longrightarrow \Omega^1_d E \underset{\iota^1_{d,E}}{\overset{\widetilde{\nabla}}{\rightleftarrows}} J^1_d E \xrightarrow[\pi^{1,0}_{d,E}]{} E \longrightarrow 0$$

the jet lift $\widetilde{\nabla}$ is a retraction of $\iota^1_{d,E}$, or equivalently, a splitting of the first jet exact sequence in ${}_A\mathrm{Mod}$.

Higher order connections

Definition

Let E be in ${}_A\mathrm{Mod}$. A (left) n-connection on E is a section $C^n\colon J^{n-1}_d E \hookrightarrow J^n_d E$ in ${}_A\mathrm{Mod}$ of the jet projection $\pi^{n,n-1}_{d,E}\colon J^n_d E \to J^{n-1}_d E$.

The n-connections are in bijective correspondence with right splittings in ${}_A\mathrm{Mod}$:

$$0 \longrightarrow S^n_d E \xhookrightarrow{\iota^n_{d,E}} J^n_d E \underset{C^n}{\overset{\pi^{n,n-1}_{d,E}}{\rightleftarrows}} J^{n-1}_d E \longrightarrow 0.$$

- n'th order differential operators $\nabla^n\colon E \to S^n_d E$ with restriction symbol the identity.
- Left splittings $\widetilde{\nabla}^n$.

Higher Order Connections $\Leftrightarrow$ Left Connections on Jets

Theorem

Assume that Ω^1_d and Ω^2_d are flat in Mod_A. Then there is a bijective correspondence between n-connections C^n on E, and left connections ∇ on $J^{n-1}_d E$ satisfying

1. $\Omega^1_d(\pi^{n-1,n-2}_{d,E}) \circ \nabla = \mathcal{S}^{n-1,0}_{d,E}$;
2. *The curvature $R_\nabla \colon J^{n-1}_d E \to \Omega^2_d J^{n-1}_d E$ has values in $\Omega^2_d S^{n-1}_d E$.*

The correspondence:

- $C^n \mapsto \nabla := \mathcal{S}^{n,0}_{d,E} \circ C^n$
- $\nabla \mapsto C^n$ *associated to ∇^n characterized by*

$$\Omega^1_d(\iota^{n-1}_{d,E}) \circ \iota^n_{\wedge,E} \circ \nabla^n = \nabla \circ j^{n-1}_{d,E} \colon E \longrightarrow \Omega^1_d J^{n-1}_d E.$$

The Symbol Sequence

$$0 \longrightarrow \mathrm{Diff}_d^{n-1}(E,F) \hookrightarrow \mathrm{Diff}_d^n(E,F) \underset{\varsigma^n_{d,E,F}}{\longrightarrow} \mathrm{Symb}_d^n(E,F) \longrightarrow 0, \qquad q^n : \mathrm{Symb}_d^n(E,F) \to \mathrm{Diff}_d^n(E,F)$$

Definition

An n-quantization for (E,F) is an ${}_A\mathrm{Hom}(F,F)$-linear right splitting of $\varsigma^n_{d,E,F}$, i.e.

$$q^n\colon \mathrm{Symb}_d^n(E,F) \longrightarrow \mathrm{Diff}_d^n(E,F).$$

such that $\varsigma^n_{d,E,F} \circ q^n = \mathrm{id}_{\mathrm{Symb}_d^n(E,F)}$. If it exists, the map

$$q = \sum_{n\in\mathbb{N}} q^n\colon \mathrm{Symb}_d(E,F) \longrightarrow \mathrm{Diff}_d(E,F)$$

is called a (full) quantization for (E,F).

Quantizations ⇔ Higher Order Connections

Theorem

Let $E \in {}_A\mathrm{Mod}$. Natural n-quantizations $q^n \colon \mathrm{Symb}^n_d(E,-) \to \mathrm{Diff}^n_d(E,-)$ are in bijective correspondence with n-connections $C^n \colon J^{n-1}_d E \to J^n_d E$. Explicitly, the correspondence is as follows

$$q^n(\varsigma^n_d(\Delta)) = r^n_{d,E,S^n_d E}(\varsigma^n_d(\Delta)) \circ \nabla^n.$$

Theorem (Full quantization)

Let E in ${}_A\mathrm{Mod}$. Suppose we have a family of connections $\nabla^{S^n_d E}$ on $S^n_d E$ and left splittings $\mathsf{s}^{1,n}$ for $\iota^n_{\wedge,E} \colon S^n_d E \to \Omega^1_d S^{n-1}_d E$. Then there is an induced full quantization q.

$$q^n(\varsigma^n_d(\Delta)) = r^n_d(\varsigma^n_d(\Delta)) \circ \mathsf{s}^{1,n-1} \circ \nabla^{S^{n-1}_d E} \circ \mathsf{s}^{1,n-2} \circ \nabla^{S^{n-2}_d E} \circ \cdots \circ \mathsf{s}^{1,1} \circ \nabla^{S^1_d E} \circ \nabla^E.$$

Definition

Let Δ be a linear differential operator of order n. Let $\Delta^{(n)} = \Delta$, and recursively define

$$\Delta^{(k)} = \Delta^{(k+1)} - q^{k+1}(\varsigma_d^{k+1}(\Delta^{(k+1)}))$$

for $0 \leq k \leq n-1$.

$$\varsigma_q(\Delta) = \varsigma_d^n(\Delta^{(n)}) + \varsigma_d^{n-1}(\Delta^{(n-1)}) + \cdots + \varsigma_d^0(\Delta^{(0)})$$

is called the total symbol of Δ with respect to the quantization q.

Proposition

$$\Delta = \sum_{i=0}^{n} q_i \circ \varsigma_d^i(\Delta^{(i)}) = q \circ \varsigma_q(\Delta),$$

Definition

Let $\hbar$ be a formal parameter. We define the total Hamiltonian map $\mathrm{Ham}_\hbar\colon \mathrm{Diff}_d(A, A) \to \mathrm{Symb}_d(A, A)$ *by*

$$\mathrm{Ham}_\hbar(\Delta) = \sum_k \hbar^{-k} \varsigma_d^k(\Delta^{(k)}),$$

where $\Delta^{(k)}$ *is as in the total symbol coming from* q*. This admits a section, given by* $h_q = \bigoplus_k \hbar^k q_k$*. The star product* $\star$ *corresponding to the quantization* q *is then given by the formula*

$$a \star b = \mathrm{Ham}_\hbar(h_q(a) \circ h_q(b)).$$

for two arbitrary elements $a, b \in \mathrm{Symb}_d(A, A)$*.*

Phase space quantization

Proposition

The star product gives a family of unital associative algebra structures on $\mathrm{Symb}_d(A, A)$, *which can be written as*

$$a \star b = \varsigma_d^{n+m}(q_n(a) \circ q_m(b))^{(n+m)}) + \hbar \varsigma_d^{n+m-1}(q_n(a) \circ q_m(b))^{(n+m-1)}) + \cdots + \hbar^{n+m} \varsigma_d^{0}(q_n(a) \circ q_m(b))^{(0)}),$$

for elements $a \in \mathrm{Symb}_d^n(A, A)$ *and* $b \in \mathrm{Symb}_d^m(A, A)$. *These new algebra structures are filtered deformations of the usual graded product on* $\mathrm{Symb}_d(A, A)$, *meaning that*

$$ab - a \star b \in \bigoplus_{k=0}^{m+n-1} \mathrm{Symb}_d^k(A, A),$$

i.e. the two products agree on the degree $m + n$ *term and*

$$a \star b = ab + \mathcal{O}(\hbar).$$

Consider the algebra $\mathbb{H}$ of quaternions, with structure equations

$$dk = -jdi + idj$$

The jet modules $J^n_d\mathbb{H}$ are

$$J^1_d\mathbb{H} \simeq \mathbb{H}^3 \quad J^2_d\mathbb{H} \simeq \mathbb{H}^4 \quad J^3_d\mathbb{H} = J^2_d\mathbb{H}, \cdots$$

We have that $\mathrm{Diff}_d(\mathbb{H}, \mathbb{H})$ is generated by ∂_i, ∂_j and R_i, R_j, R_k.

Proposition

There is a unique bimodule connection ∇ on Ω^1_d, with generalized braiding σ given by $A \otimes_{\mathbb{H}} B \mapsto -B \otimes_{\mathbb{H}} A$ for $A, B \in \{di, dj\}$ and extended bilinearly. This ∇ is torsion free, and is the Grassmann connection for the frame di, dj.

Proposition

Let $L_k = L_k^{(2)} + L_k^{(1)} + L_k^{(0)}$ *be the decomposition of* L_k *given by the canonical quantization from the theorem, taking* ∇ *as the connection on* $\Omega^1_d = S^1_d$, $\mathsf{s}^{1,1} = \frac{1}{2}(\mathrm{id} + \sigma)$, *and* $\widetilde{\nabla}^1 = \widetilde{d}$. *Then*

$$L_k^{(2)} = 2[\partial_i, \partial_j]$$
$$L_k^{(1)} = 2(\partial_i \cdot j - \partial_j \cdot i)$$
$$L_k^{(0)} = R_k$$

Finally let us describe the star product on $\mathrm{Symb}_d(\mathbb{H},\mathbb{H})$. We will write it in terms of a generator set, letting $x_i = [R_i]$, $x_j = [R_j]$ and $p_i = [\partial_i]$, $p_j = [\partial_j]$ play the rôles of generalized position and momenta.

Proposition

The star product on $\mathrm{Symb}_d(\mathbb{H},\mathbb{H})$ *defined by the quantization coming from* ∇ *is given by*

$$\begin{aligned} x_a \star x_b &= x_a x_b \\ x_a \star p_b &= x_a p_b \\ p_a \star x_b &= -x_b p_a + \hbar \delta^a_b \\ p_a \star p_b &= p_a p_b \end{aligned}$$

where $a, b \in \{i, j\}$, *and* δ^a_b *is the Kronecker symbol. In particular, for all values of* $\hbar$ *we have that* x_i *and* x_j *generate a subalgebra isomorphic to* $\mathbb{H}^{op}$, *and* $p_i p_j = -p_j p_i$, $p_i^2 = p_j^2 = 0$. *The original symbol algebra structure is recovered for* $\hbar = 0$.