Conserved quantities for conformal loxodromes on conformal sphere

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Conformal sphere S^n

A conformal Riemannian structure on a smooth manifold M of dimension n ... class of Riemannian metrics that differ by a multiple of an everywhere positive function.

- The homogeneous model (maximally symmetric conformal manifold) ... the sphere S^n with the conformal class represented by the standard round metric.
- The Lie group of conformal transformations ... isomorphic to $G := O(n+1, 1)$, with the Lie algebra $\mathfrak{g} := \mathfrak{so}(n+1, 1)$.
- ... follows from the realization of S^n as the projectivization of the cone of non-zero null-vectors in the pseudo-Euclidean space $\mathbb{R}^{n+1,1}$ of signature $(n+1, 1)$.
- The group G ... acts transitively on S^n and the stabilizer of a point is the Poincaré subgroup $P \subset G$, so $S^n \cong G/P$.

Matrix description of S^n

We write elements of $\mathfrak{g}$ as $(1, n, 1)$-block matrices

$$\begin{pmatrix} a & Z & 0 \\ X & A & -Z^T \\ 0 & -X^T & -A \end{pmatrix}$$

for $a \in \mathbb{R}$, $A \in \mathfrak{so}(n)$ and $X, Z^t \in \mathbb{R}^n$.

- The Lie algebra $\mathfrak{p} \subset \mathfrak{g}$ corresponds to the upper triangular matrices ... red
- There is a natural complement $\mathfrak{c}$ of $\mathfrak{p}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{p}$ giving natural exponential coordinates on S^n around the origin $o = eP$ ... blue

... reflects the projectivization of the cone in $\mathbb{R}^{n+1,1}$

$$\exp \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^T & 0 \end{pmatrix} o = \begin{pmatrix} 1 & 0 & 0 \\ X & E & 0 \\ -\frac{1}{2}\langle X,X\rangle & -X^T & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ X \\ -\frac{1}{2}\langle X,X\rangle \end{pmatrix},$$

$\langle\, , \,\rangle$ ... the standard scalar product on $\mathbb{R}^n$

Curves

We consider the curve

$$\gamma = (\gamma_i(t)), \quad i = 1, \dots, n.$$

We denote

- $U = \gamma' = (\frac{d}{dt}\gamma(t))$ ... velocity vector
- $A = U' = (\frac{d^2}{dt^2}\gamma(t))$ ... acceleration vector

and so on for subsequent derivatives.

We denote the length of the velocity vector

$$u = \sqrt{\langle U, U \rangle}$$

for the standard scalar product on $\mathbb{R}^n$.

Standard tractors

We employ tractor viewpoint on curves here.

- We consider the standard tractor bundle with the tractor metric ... modeled on $\mathbb{R}^{n+2}$ with the standard pseudo-metric of signature $(n+1, n)$.
- We deal with the standard representation

$$\rho : \mathfrak{so}(n+1,1) \to \mathfrak{gl}(\mathbb{R}^{n+2}).$$

- The (flat) standard tractor connection ... decomposes into the fundamental derivative and the algebraic action

$$D + \rho,$$

where we view tractors as sections of $\mathfrak{c} \times \mathbb{R}^{n+2}$, i.e. we employ the description in exponential coordinates ... restriction to $\mathfrak{c}$.

Tractor series for curves

For each curve γ ... canonical series of derived standard tractors by means of the tractor derivative.
We write these tractors via coordinates in the standard basis of $\mathbb{R}^{n+2}$ that we decompose according to the standard tractors as

$$\mathbb{R}^{1+n+1} = \langle e_0 \rangle + \langle e_1, \ldots, e_n \rangle + \langle e_{n+1} \rangle.$$

$$\mathbb{T} = u^{-1} e_0$$

$$\mathbb{U} = \mathbb{T}' = -u^{-3} \langle U, A \rangle e_0 + \sum_{i=1}^{n} u^{-1} U_i e_i$$

$$\mathbb{A} = \mathbb{U}' = (u^{-3}(\langle A, A \rangle - \langle U, A' \rangle) + 3u^{-5} \langle U, A \rangle^2) e_0$$
$$+ \sum_{i=1}^{n} (-2u^{-3} \langle U, A \rangle U_i + u^{-1} A_i) e_i - u e_{n+1}$$

$$\ldots$$

Conformal circles

The curve is a conformal circle ... $\mathbb{T}, \mathbb{U}, \mathbb{A}, \mathbb{A}'$ are linearly dependent ... $T_3 := \mathbb{T} \wedge \mathbb{U} \wedge \mathbb{A}$ is parallel along γ for the tractor connection.

- For each (parametrized) curve ... by means of the tractor metric

	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{A}$	$\mathbb{A}'$
$\mathbb{T}$	0	0	-1	0
$\mathbb{U}$	0	1	0	$-\alpha_1$
$\mathbb{A}$	-1	0	α_1	$\frac{1}{2}\alpha_1'$
$\mathbb{A}'$	0	$-\alpha_1$	$\frac{1}{2}\alpha_1'$	α_2

- Then we can use e.g. the Cramer rule to write the tractor combination explicitly

$$\mathbb{A}' = -\alpha_1 \mathbb{U} - \frac{1}{2}\alpha_1' \mathbb{T}$$

- Then we compute

$$(\mathbb{T}\wedge\mathbb{U}\wedge\mathbb{A})' = \mathbb{U}\wedge\mathbb{U}\wedge\mathbb{A} + \mathbb{T}\wedge\mathbb{A}\wedge\mathbb{A} + \mathbb{T}\wedge\mathbb{U}\wedge(-\alpha_1\mathbb{U} - \frac{1}{2}\alpha_1'\mathbb{T}) = 0.$$

3-tractor T_3

We compute $T_3 := \mathbb{T} \wedge \mathbb{U} \wedge \mathbb{A} \in \wedge^3 \mathbb{R}^{n+2}$ explicitly in the basis $b_3 := \{e_i \wedge e_j \wedge e_k \ : \ i < j < k\}$

$$T_3 = u^{-3} \sum_{i<j} (U_i A_j - U_j A_i)\, e_0 \wedge e_i \wedge e_j - u^{-1} \sum_i U_i\, e_0 \wedge e_i \wedge e_{n+1}.$$

Denoting ϵ_{ij} the alternating symbol and omitting the sums ...

$$T_3 = u^{-3} \epsilon_{ij} U_i A_j\, e_0 \wedge e_i \wedge e_j - u^{-1} U_i\, e_0 \wedge e_i \wedge e_{n+1}.$$

On conformal sphere S^n, each conformal Killing-Yano 2-tensor corresponds to an element of the tractor space $\wedge^3 \mathbb{R}^{n+2}$. Then its pairing (w.r.t. the tractor metric) with the 3-tractor $\mathbb{T} \wedge \mathbb{U} \wedge \mathbb{A}$ gives a conserved quantity of conformal circles.

CKY 2-forms and tractors

CKY 2-form ... the tractor bundle for the representation

$$\rho_3 : \mathfrak{g} \to \mathfrak{gl}(\wedge^3 \mathbb{R}^{n+2})$$

- To describe the corresponding tractors in exponential coordinates ...

$$\exp(X) \mapsto \exp(-\rho_3(X))(w)$$

  where $X \in \mathfrak{c}$ reflects exponential coordinates on the sphere and w are coordinates in the respresentation space.

- Thus we describe the standard action of

$$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^T & 0 \end{pmatrix}$$

  on the standard basis of $\mathbb{R}^{n+2}$ and extend tensorialy.

Conserved quantities for circles

We will use the notation

$$x = (x_i), \quad U = (y_i) = y, \quad A = (z_i) = z$$

where $i = 1, \ldots, n$. Here $N = n + 1$ and $0 < i < j < N$ and ϵ is the alternating symbol.

$$Q_{0iN} = \frac{1}{\|y\|} y_i + \frac{1}{\|y\|^3} \sum_{k \neq i} x_k \epsilon_{ik} y_i z_k$$

$$Q_{0ij} = \frac{1}{\|y\|} \epsilon_{ji} x_j y_i + \frac{1}{2\|y\|^3} (-\sum_{k=1}^{n} x_k^2 + 2x_i^2 + 2x_j^2) \epsilon_{ij} y_i z_j +$$
$$\frac{1}{\|y\|^3} \sum_{k \neq i,j} x_k (x_i \epsilon_{kj} y_k z_j + x_j \epsilon_{ik} y_i z_k)$$

$$Q_{0ij} = \frac{1}{\|y\|^3} \epsilon_{ij} y_i z_j \qquad Q_{ijk} = \frac{1}{\|y\|^3} \epsilon_{ijk} x_i y_j z_k$$

Dimension 3

$$Q_{014} = \frac{y_1}{\sqrt{y_1{}^2 + y_2{}^2 + y_3{}^2}} + \frac{x_2\left(y_1 z_2 - y_2 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} + \frac{x_3\left(y_1 z_3 - y_3 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{024} = \frac{y_2}{\sqrt{y_1{}^2 + y_2{}^2 + y_3{}^2}} - \frac{x_1\left(y_1 z_2 - y_2 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} + \frac{x_3\left(y_2 z_3 - y_3 z_2\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{034} = \frac{y_3}{\sqrt{y_1{}^2 + y_2{}^2 + y_3{}^2}} - \frac{x_1\left(y_1 z_3 - y_3 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} - \frac{x_2\left(y_2 z_3 - y_3 z_2\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{012} = -\frac{x_1 y_2 - x_2 y_1}{\sqrt{y_1{}^2 + y_2{}^2 + y_3{}^2}} + \frac{\left(x_1{}^2 + x_2{}^2 - x_3{}^2\right)\left(y_1 z_2 - y_2 z_1\right)}{2\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} - \frac{x_1 x_3\left(y_2 z_3 - y_3 z_2\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} + \frac{x_2 x_3\left(y_1 z_3 - y_3 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{013} = -\frac{x_1 y_3 - x_3 y_1}{\sqrt{y_1{}^2 + y_2{}^2 + y_3{}^2}} + \frac{\left(x_1{}^2 - x_2{}^2 + x_3{}^2\right)\left(y_1 z_3 - y_3 z_1\right)}{2\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} + \frac{x_1 x_2\left(y_2 z_3 - y_3 z_2\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} + \frac{x_2 x_3\left(y_1 z_2 - y_2 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{023} = -\frac{x_2 y_3 - x_3 y_2}{\sqrt{y_1{}^2 + y_2{}^2 + y_3{}^2}} + \frac{\left(-x_1{}^2 + x_2{}^2 + x_3{}^2\right)\left(y_2 z_3 - y_3 z_2\right)}{2\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} + \frac{x_1 x_2\left(y_1 z_3 - y_3 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}} - \frac{x_1 x_3\left(y_1 z_2 - y_2 z_1\right)}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{124} = \frac{y_1 z_2 - y_2 z_1}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{134} = \frac{y_1 z_3 - y_3 z_1}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{234} = \frac{y_2 z_3 - y_3 z_2}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

$$Q_{123} = \frac{x_1 y_2 z_3 - x_1 y_3 z_2 - x_2 y_1 z_3 + x_2 y_3 z_1 + x_3 y_1 z_2 - x_3 y_2 z_1}{\left(y_1{}^2 + y_2{}^2 + y_3{}^2\right)^{3/2}}$$

Loxodromes and more general curves

- The curve is circle … dependency of $\mathbb{T}, \mathbb{U}, \mathbb{A}, \mathbb{A}'$ … determinant Δ_4 of the table viewed as a matrix satisfies $\Delta_4 = 0$
- The curve satisfies $\Delta_5 = 0$ … tractors $\mathbb{T}, \mathbb{U}, \mathbb{A}, \mathbb{A}', \mathbb{A}''$ are linearly dependent … more general family of curves than loxodromes.

We then compute

$$\mathbb{A}'' = \frac{\Delta_4'}{2\Delta_4}\mathbb{A}' - \alpha_1\mathbb{A} + \frac{1}{2\Delta_4}(\alpha_1'(2\alpha_2 - \Delta_4) - \alpha_1\alpha_2')\mathbb{U} + \frac{1}{4\Delta_4}(2\alpha_1(\alpha_1')^2 - 4\Delta_4^2 - 2\alpha_2''\Delta_4 - \alpha_1'\alpha_2')\mathbb{T}$$

Let us naively derive $\Delta_4^a \cdot \mathbb{T} \wedge \mathbb{U} \wedge \mathbb{A} \wedge \mathbb{A}'$ for $a \in \mathbb{R}$

$$(\Delta_4^a \cdot \mathbb{T} \wedge \mathbb{U} \wedge \mathbb{A} \wedge \mathbb{A}')' = ((\Delta_4^a)' + \Delta_4^a \cdot \frac{\Delta_4'}{2\Delta_4}) \cdot \mathbb{T} \wedge \mathbb{U} \wedge \mathbb{A} \wedge \mathbb{A}'$$

- $a\Delta_4^{a-1}\Delta_4' + \Delta_4^a \cdot \frac{\Delta_4'}{2\Delta_4} = 0$ … parallel for $a = -\frac{1}{2}$; $\Delta_4' = 0$

Δs … relative invariants, (non)vanishing independent of reparametrization

Loxodromes

There is a series of absolute invariants

$$\kappa_1 = -\frac{1}{2}(-\Delta_4)^{-\frac{5}{2}}(\alpha_1\Delta_4^2 - \frac{1}{2}\Delta_4\Delta_4'' + \frac{9}{16}(\Delta_4')^2)$$
$$\kappa_2 = -(-\Delta_4)^{-\frac{1}{4}}((-1)\Delta_5)^{\frac{1}{2}}\Delta_4^{-1}$$
$$\dots$$
$$\kappa_\ell = -(\Delta_4)^{-\frac{1}{4}}(\Delta_{\ell+1}\Delta_{\ell+3})^{\frac{1}{2}}\Delta_{\ell+2}^{-1}$$

- The curve is a loxodrome ... κ_1 is constant and κ_2 vanishes.

Then

- The condition $\kappa_2 = 0$ is equivalent to $\Delta_5 = 0$ for $\Delta_4 \neq 0$.
- There is a question on additional conditions to the dependency of tractors that emphasize loxodromes.
- Assuming Δ_4 constant gives α_1 constant for κ_1 constant.
- The condition α_1 constant gives a parametrization.
- If Δ_4 and α_1 are constant, then κ_1 is constant.

4-tractor T_4

We can analogously use $T_4 := \mathbb{T} \wedge \mathbb{U} \wedge \mathbb{A} \wedge \mathbb{A}'$ and its pairing with elements of $\wedge^4 \mathbb{R}^{n+2}$ (that correspond to CKY 3-forms) to find conserved quantities.

We compute T_4 in coordinates in the basis

$$b_4 = \{e_i \wedge e_j \wedge e_k \wedge e_l \ : \ i < j < k < l\}$$

and we get

$$T_4 = u^{-4} \epsilon_{ijk} U_i A_j A'_k \, e_0 \wedge e_i \wedge e_j \wedge e_k -$$
$$3u^{-4} \langle U, A \rangle \epsilon_{ij} U_i A_j \, e_0 \wedge e_i \wedge e_j \wedge e_{n+1} +$$
$$u^{-2} \epsilon_{ij} U_i A'_j \, e_0 \wedge e_i \wedge e_j \wedge e_{n+1}$$

Conserved quantities

We use the notation

$$x = (x_i), \quad U = (y_i) = y, \quad A = (z_i) = z, \quad A' = (v_i) = v$$

where $i = 1, \dots, n$. Here $N = n + 1$ and $0 < i < j < k < l < N$.

$$Q_{0ijN} = \frac{3}{\|y\|^4}\langle y, z\rangle \epsilon_{ij} y_i z_j - \frac{1}{\|y\|^2}\epsilon_{ij} y_i v_j + \frac{1}{\|y\|^4}\sum_{i,j\neq l} x_l \epsilon_{ijl} y_i z_j v_l$$

$$Q_{0ijk} = \frac{3}{\|y\|^4}\langle y, z\rangle \epsilon_{ijk} x_i y_j z_k - \frac{2}{\|y\|^2}\epsilon_{ijk} x_i y) j v_k -$$

$$\frac{1}{2\|y\|^4}(\sum_{l=1}^{n} x_l^2 - 2x_i^2 - 2x_j^2 - 2x_k^2)\epsilon_{ijk} +$$

$$\sum_{l\neq i,j,k} x_l (x_i \epsilon_{ljk} y_j z_k v_l + x_j \epsilon_{ilk} y_i z_k v_l + x_k \epsilon_{ijl} y_i z_j v_l)$$

$$Q_{ijkN} = \frac{1}{\|y\|^4}\epsilon_{ijk} y_i z_j v_k \qquad Q_{ijkl} = \frac{1}{\|y\|^4}\epsilon_{ijkl} x_i y_j z_k v_l$$

Dimension 3

... use scalar product, cross product and wedge product
... thus $a \wedge b \wedge c$ is the scale given by the determinant of the corresponding matrix and

$$a \wedge b = \left(\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, - \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}, \begin{pmatrix} a_2 & ta_3 \\ b_2 & tb_3 \end{pmatrix} \right)$$

$$Q_{0124} = \frac{1}{\|y\|^4} x_1 y \wedge z \wedge v - \frac{1}{\|y\|^2} y \wedge v + \frac{3}{\|y\|^4} \langle y, z \rangle y \wedge z$$
$$Q_{0134} = \frac{1}{\|y\|^4} x_2 y \wedge z \wedge v - \frac{1}{\|y\|^2} y \wedge v + \frac{3}{\|y\|^4} \langle y, z \rangle y \wedge z$$
$$Q_{0234} = \frac{1}{\|y\|^4} x_3 y \wedge z \wedge v - \frac{1}{\|y\|^2} y \wedge v + \frac{3}{\|y\|^4} \langle y, z \rangle y \wedge z$$
$$Q_{0123} = \frac{3}{\|y\|^4} \langle y, z \rangle x \wedge y \wedge z - \frac{1}{\|y\|^2} x \wedge y \wedge v + \frac{1}{2} \frac{\|x\|^2}{\|y\|^4} y \wedge z \wedge v$$
$$Q_{1234} = -\frac{1}{\|y\|^4} y \wedge z \wedge v$$

Dunajski-Krynski, 2021

Consider 4^{th} order ODE

$$\frac{dC}{dt} = 0$$

$$C = u^{-2}(\dot{A} - u^{-2}\langle A, A\rangle - 2u^{-2}\langle A, U\rangle +$$

$$4u^{-4}\langle U, A\rangle^2 - 2u^{-2}\langle \dot{A}, U\rangle)$$

...equivalent to the system od four 1^{st} order ODE ...
$4n$-dimensional Hamiltonian phase space variables $(X, U, \mathcal{P}, \mathcal{R})$

$$\dot{X} = U$$

$$\dot{U} = u^2\mathcal{R} - 2\langle U, \mathcal{R}\rangle U$$

$$\dot{\mathcal{P}} = 0$$

$$\dot{\mathcal{R}} = -|\mathcal{R}|^2 U + 2\langle U, \mathcal{R}\rangle\mathcal{R} - \mathcal{P}$$

... Hamiltonian system for the Hamiltonian

$$H = \frac{1}{2}u^2|\mathcal{R}|^2 - \langle U, \mathcal{R}\rangle\mathcal{R} - \mathcal{P}$$

Black magic from Prim in dim 3

- ... uses Lagrangian viewpoint and conformal Killing vector fields to find conserved quantities of the Mercator equation

Let V be a conformal Killing vector field restricted to γ. Then the function

$$F = \frac{d}{dt}\langle W, \dot{V}\rangle + \langle \dot{W}, \dot{V}\rangle - \langle C, V\rangle, \quad W = u^{-2}U$$

is constant on any solution curve of the Mercator equation.

- ... finds functionally independent set of her quantities
- ... shows that both her and mine quantities commute with the Hamiltonian
- ... finds direct relation between the quantities in the Hamiltonian viewpoint

Black magic from Prim in dim 3

$$
\begin{aligned}
F_T &= \mathcal{P} \\
F_R &= X \wedge \mathcal{P} + U \wedge \mathcal{R} \\
F_D &= < X, \mathcal{P} > + < U, \mathcal{R} > \\
F_S &= |X|^2 \mathcal{P} + 2 < X, U > \mathcal{R} - 2F_D X - 2(1 + < X, \mathcal{R} >)U \\
L_1 &= |X|^2 \det(U\,\mathcal{R}\,\mathcal{P}) - 2\det(X\,U\,\mathcal{P}) + 2 < U, \mathcal{R} > \det(X\,U\,\mathcal{R}) \\
L_{2,3,4} &= U \wedge \mathcal{P} - \det(U\,\mathcal{R}\,\mathcal{P})X - < U, \mathcal{R} > U \wedge \mathcal{R} \\
L_5 &= \det(U\,\mathcal{R}\,\mathcal{P}) \\
H &= \frac{1}{2}|U|^2|\mathcal{R}|^2 - < U, \mathcal{R} >^2 + < U, \mathcal{P} >,
\end{aligned}
$$

$$
\begin{aligned}
L_1 &= - < F_R, F_S > \\
L_{2,3,4} &= \frac{1}{2} F_T \wedge F_S - F_D F_R \\
L_5 &= < F_T, F_R > \\
H &= \frac{1}{2}\left(|F_R|^2 - < F_T, F_S > - F_D^2 \right),
\end{aligned}
$$

Note on equations

- The invariant part of the tractor equation ... an equation for curves satisfying $\Delta_5 = 0$
- Assuming α and Δ_4 are constants ... the two equations coincide up to a multiple
- ... generally they are different
- ... what is a minimal condition to assume to get the equations *same up to multiple*