Spans of Quantum L_∞ Algebras

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Theorem: If C is a cartesian category with pullbacks, Span(C) is a dagger compact category.

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For more about this, see [Abramsky-Coecke'04,08], [Selinger'07] and the webpage of John Baez.

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Plan: Do this for gauge QFTs/SFTs in Batalin-Vilkovisky formalism!

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$$\Delta = \frac{1}{2}(-1)^{|i|}\omega^{ij}\frac{\partial^2}{\partial\phi^i\partial\phi^j}.$$

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Def:[Zwiebach'92] A **quantum** L_∞ **algebra** on V is a formal series

$$S = \sum_{\substack{n\geq 2, g\geq 0\\ 2g+n\geq 1}} S_n^g \hbar^g \in \mathcal{F}V \equiv \widehat{\mathrm{Sym}}(V^*)((\hbar)), \qquad \text{st.} \qquad \Delta e^{S/\hbar} = 0.$$

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or the algebra over the operad $\mathsf{F}(\mathrm{Mod}(\mathsf{Lie}^!))$, see [Markl'97].

Let $H \coloneqq \mathrm{Coh}(Q : V \to V)$ induced by $\{S_{\text{free}}, -\} \equiv \{S_2^0, -\} : V^* \to V^*$,

$$V = H \oplus \operatorname{Im} Q \oplus I_{GF},$$

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Prop: There is a quantum L_∞ algebra structure $W \in \mathcal{F}H$ on cohomology called the **minimal model** (or *effective action*) given (heuristically) by

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Def: We say quantum L_∞ algebras S and S' are **quasi-isomorphic** if their minimal models W and W' are isomorphic. This means there is an invertible **non-linear morphism of quantum L_∞ algebras**,

$$\phi : H \xrightarrow{\cong} H' \quad \text{(defined by its action on } \phi^* : \mathcal{F}H' \to \mathcal{F}H),$$

$$\phi^*\{-,-\}' = \{\phi^* -, \phi^* -\},$$

$$\phi^* \circ (\hbar\Delta' + \{W', -\}') = (\hbar\Delta + \{W, -\}) \circ \phi^*.$$

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$$L = \{(h + v, h) \in V \times H \mid v \in I_{GF}\} \subset \overline{V} \times H.$$

Linear Lagrangian Relations

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- For *linearly* quasi-iso S and S', $\exists$ the linear Lagrangian relation

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which *relates* S and S' in the sense that $\phi^* \int_L e^{S/\hbar} = \int_{L'} e^{S'/\hbar}$.

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Lesson: *A lot can be encoded in linear relations, but there are limitations.* For more details, see [Jurčo-Pulmann-Z'24].

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Theorem:[Doubek-Jurčo-Pulmann'17] There is a non-linear isomorphism (identity modulo $\hbar$) of quantum L_∞ algebras $\psi_V : V \to V$ between S and

$$p^*(W) + S_{\text{free}} \in \mathcal{F}V = \mathcal{F}(H \oplus \operatorname{Im} Q \oplus I_{GF}),$$

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Remark: This comes from a homotopy in the sense of [Costello'11], given by a quantum L_∞ structure on $\mathcal{F}V \otimes \Omega^\bullet([0,1])$. There is an alternative approach, *abstract homological perturbation lemma* [Chuang-Lazarev'17].

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- Then we have the following diagram of quantum L_∞ morphisms.

$$\begin{tikzcd}
(V, S_{\text{free}} + p^*W) \arrow[r, "\psi_V"] \arrow[dr, "p"'] & (V, S) \arrow[d, dashed, "L"] & (V', S') \arrow[d, dashed, "L'"'] & (V', S'_{\text{free}} + (p')^*W') \arrow[l, "\psi_{V'}"'] \arrow[dl, "p'"] \\
& (H, W) \arrow[r, "\phi"'] & (H', W') &
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Now the pullback in GrVect (precisely dg (-1)-symplectic vector spaces with Poisson chain maps) induces a diagram of quantum L_∞ algebra morphisms!

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With

$$
\begin{aligned}
X &:= H \oplus (\operatorname{Im} Q \oplus I_{GF}) \oplus (\operatorname{Im} Q' \oplus I'_{GF}), \\
S_X &:= (p \circ \pi)^*(W) + \pi^* S_{\text{free}} + (\pi')^* S'_{\text{free}}.
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Now we can:

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Spans of Quantum L_∞ Algebras (And?)

Now we can:

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- Turn this into a *span of relations* $\widetilde{L}, \widetilde{L}'$.

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- Consider a non-linear quasi-isomorphism ϕ.

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- Using Lagrangian relations or spans, we can describe a *homotopy* between quantum L_∞ structures on *non-isomorphic vector spaces*. Can we extend this approach to the full simplicial enrichment following [Costello'11]?

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Thank you for your attention!