

Applications of CFT determinant formulas in number theory

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Introduction

Computations [Kac, Zhu, Dong-Li-Mason, Mason-Tuite-Z, Tuite-Z] related to vertex algebras were always a source of new identities in number theory. While finding closed expressions for vertex operator algebra characters, we finally obtain relations among modular forms, fundamental kernels, and q -series.

Various ways to compute characters for vertex operator algebra modules lead to generation of modular forms as well as interesting identities for them in terms of elliptic functions. In particular, [Mason-Tuite-Z] considerations of the twisted partition function on the torus for the rank two free fermion vertex operator superalgebra allow us to provide a pure algebraic explanation of Jacobi triple product identity [Kac].

At the same time, computations of higher correlation functions on a genus one Riemann surface provide us [Mason-Tuite-Z] with an elliptic version of the Fay's trisecant identity [Fay] known from algebraic geometry. Various identities for powers of the η -function appear [Kac] in studies of affine Lie algebras. Those are important in number theory.

Torus correlation functions

For an automorphism g twisted module $\mathcal{V}$ for a vertex operator algebra V we find closed formulas for correlation functions of vertex operators $\mathcal{Y}$ on the torus, $q = e^{2\pi i\tau}$, with local coordinates z_i , $v_i \in V$, $1 \leq i \leq n$, [Mason-Tuite-Z]:

$$Z_V^{(1)} \left[\begin{array}{c} f \\ g \end{array} \right] (v_1, z_1, \dots, v_n, z_n; q) = \text{Tr}_{\mathcal{V}} \left(f \mathcal{Y}(v_1, z_1) \dots \mathcal{Y}(v_n, z_n) q^{L(0) - C/24} \right),$$

where $L(0)$ is the Virasoro algebra generator, and C is central charge.

The formal parameter is associated to a complex parameter on the torus. Final expressions are given by determinants of matrices with elements being coefficients in the expansions of the regular parts of corresponding differentials: Bergman (bosons) or Szegő (fermions) kernels [Mason-Tuite-Z].

In this talk we derive some new genus two generalizations of the fundamental formulas for powers of the η -function in terms of deformed versions [Dong-Li-Mason, Mason-Tuite-Z] of Weierstrass functions and Eisenstein series.

In particular, we find that powers of the modular discriminant are expressed (up to theta-functions multipliers) via determinants of finite matrices containing combinations of deformed modular functions. In the proof we use the generalized elliptic version of the Fay's trisecant identity for a vertex operator superalgebra.

Modular discriminant and Eisenstein series

The modular discriminant is defined by $\Delta(\tau) = \eta(\tau)^{24}$, where $\eta(\tau)$ is the Dedekind eta-function, $q = e^{2\pi i\tau}$, $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$.

Recall $E_n(\tau)$ is equal to 0 for n odd, and for n even is the Eisenstein series [Serre]

$$E_n(\tau) = -\frac{B_n(0)}{n!} + \frac{2}{(n-1)!} \sum_{r \geq 1} \frac{r^{n-1} q^r}{1 - q^r},$$

where $B_n(0)$ is the n -th Bernoulli number

$$\frac{q_z^\lambda}{q_z - 1} = \frac{1}{z} + \sum_{n \geq 1} \frac{B_n(\lambda)}{n!} z^{n-1}.$$

One finds [Ramanujan] the relations: $E_8 = E_4^2$, $E_{10} = E_4 E_6$,
 $E_{12} = \frac{441}{691} E_4^3 + \frac{250}{691} E_6^2$.

Classical Gavan formula:

Then the fundamental classical formulas for the modular discriminant follow

$$\Delta(\tau) = \frac{1}{1728} (E_4^3(\tau) - E_6^2(\tau)) = \frac{1}{1728} \det \begin{pmatrix} \sqrt{7/3}E_4(\tau) & E_6(\tau) \\ E_6(\tau) & \sqrt{7/3}E_8(\tau) \end{pmatrix}.$$

The next formula is due to F. Garvan:

$$\Delta^2(\tau) = -\frac{691}{250 (1728)^2} \det \begin{pmatrix} E_4(\tau) & E_6(\tau) & E_8(\tau) \\ E_6(\tau) & E_8(\tau) & E_{10}(\tau) \\ E_8(\tau) & E_{10}(\tau) & E_{12}(\tau) \end{pmatrix},$$

which was then proved and generalized in [Milne]. In this talk we give various generalizations for higher powers of the modular discriminant computed as a determinant of matrices containing deformed Weierstrass functions [Dong-Li-Mason, Mason-Tuite-Z].

The generalized Garvan formulas

Computations of the twisted partition function $Z_V^{(1)} \left[\begin{smallmatrix} f \\ g \end{smallmatrix} \right] (\tau)$ for the free fermion vertex operator superalgebra leads to two alternative expressions (see, e.g., [Kac, Mason-Tuite-Z]) as expansion over a basis:

$$Z_V^{(1)} \left[\begin{smallmatrix} f \\ g \end{smallmatrix} \right] (\tau) = q^{\kappa^2/2 - 1/24} \prod_{l \geq 1} (1 - \theta^{-1} q^{l - \frac{1}{2} - \kappa}) (1 - \theta q^{l - \frac{1}{2} + \kappa}),$$

and

$$Z_V^{(1)} \left[\begin{smallmatrix} f \\ g \end{smallmatrix} \right] (\tau) = \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)} \vartheta^{(1)} \left[\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix} \right] (0, \tau),$$

in terms of the torus theta series with characteristics:

$$\vartheta^{(1)} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z, \tau) = \sum_{n \in \mathbb{Z}} \exp [i\pi(n+a)^2\tau + (n+a)(z + 2\pi ib)].$$

Here we define $f = e^{2\pi i \alpha a(0)}$, $g = e^{2\pi i \beta a(0)}$, with some parameters $\alpha, \beta \in \mathbb{R}$, and zero mode $a(0)$ of a Heisenberg subalgebra in the rank two free fermionic vertex operator superalgebra [Mason-Tuite-Z].

We also define $\phi = e^{-2\pi i \beta}$ and $\theta = e^{-2\pi i \alpha}$. Note that

$$Z_V^{(1)} \left[\begin{array}{c} f \\ g \end{array} \right] (\tau) = 0 \text{ for } (\theta, \phi) = (1, 1), \text{ i.e., } (\alpha, \beta) \equiv (0, 0) \pmod{\mathbb{Z}}.$$

Comparing two representations we obtain Jacobi triple product formula [Kac] which can be rewritten in the form:

$$\eta(\tau) = q^{-\kappa^2/2+1/24} e^{2\pi i(\alpha+1/2)(\beta+1/2)} \vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] (0, \tau) \det(1 - T_0),$$

where the determinant

$$\det(1 - T_0) = \left(\prod_{l \geq 1} (1 - \theta^{-1} q^{l-\frac{1}{2}-\kappa})(1 - \theta q^{l-\frac{1}{2}+\kappa}) \right)^{-1},$$

corresponds to sphere self-sewing to form a torus [Tuite-Z]. Thus we get the identity for the first power of the η -function.

Deformed Weierstrass functions

In [Dong-Li-Mason, Mason-Tuite-Z] the deformed Weierstrass functions (which can be expressed via deformed Eisenstein series) were defined and studied:

$$P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau) = - \sum'_{n \in \mathbb{Z} + \lambda} \frac{q_z^n}{1 - \theta^{-1} q^n} = \frac{1}{z} - \sum_{n \geq 1} \frac{1}{n} E_n \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) z^{n-1},$$

for $q = e^{2\pi i \tau}$, and where $\sum'$ means we omit $n = 0$ if $(\theta, \phi) = (1, 1)$.

$$\begin{aligned} E_n \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) &= -\frac{B_n(\lambda)}{n!} + \frac{1}{(n-1)!} \sum'_{r \geq 0} \frac{(r+\lambda)^{n-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} \\ &\quad + \frac{(-1)^n}{(n-1)!} \sum'_{r \geq 1} \frac{(r-\lambda)^{n-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}, \end{aligned}$$

where $\sum'$ means we omit $r = 0$ if $(\theta, \phi) = (1, 1)$.

In addition to that, for integers $m_i, n_j \geq 0$, satisfying $\sum_{i=1}^r m_i = \sum_{j=1}^s n_j$, let us introduce the notation

$$\Theta_{r,s,(m_i,n_j)}^{(1)}(x,y,\tau) = \frac{\prod_{1 \leq i \leq r, 1 \leq j \leq s} \vartheta^{(1)} \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] (x_i - y_j, \tau)^{m_i n_j}}{\prod_{1 \leq i < k \leq r} \vartheta^{(1)} \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] (x_i - x_k, \tau)^{m_i m_k} \prod_{1 \leq j < l \leq s} \vartheta^{(1)} \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] (y_j - y_l, \tau)^{n_j n_l}}.$$

Let us introduce $\mathbf{P}_n(\theta, \phi)$, a the $n \times n$ matrix:

$$\mathbf{P}_n(\theta, \phi) = \left[P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x_i - y_j, \tau) \right], \quad (1 \leq i, j \leq n),$$

and another $(n+1) \times (n+1)$ matrix $\mathbf{Q}_n$:

$$\mathbf{Q}_n = \begin{pmatrix} P_1(x_1 - y_1, \tau) & \dots & P_1(x_1 - y_n, \tau) & 1 \\ \vdots & \ddots & & \vdots \\ P_1(x_n - y_1, \tau) & & P_1(x_n - y_n, \tau) & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}.$$

Proposition

Generalizing Garvan's formula, or $(\theta, \phi) \neq (1, 1)$ one has

$$\Delta^n(\tau) = - \frac{\vartheta^{(1)} \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right] (0, \tau) \Theta_{8n, 8n, (1, 1)}^{(1)}(x, y, \tau)}{\vartheta^{(1)} \left[\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix} \right] \left(\sum_{i=1}^{8n} (x_i - y_i), \tau \right)} \det(\mathbf{P}_{8n}(\theta, \phi)),$$

for $(\theta, \phi) = (1, 1)$,

$$\Delta^n(\tau) = i \frac{\Theta_{8n+1, 8n+1, (1, 1)}^{(1)}(x, y, \tau)}{\vartheta^{(1)} \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right] \left(\sum_{i=1}^{8n+1} (x_i - y_i), \tau \right)} \det(\mathbf{Q}_{8n+1}).$$

This can be also expressed in terms of deformed Eisenstein series by substitution of the definition of $P_1 \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (z, \tau)$ in terms of $E_n \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\tau)$ leading to a quite involved formula which we do not give here.

Recall the genus one prime form $K^{(1)}(z, \tau)$ [Mumford, Fay]:

$K^{(1)}(z, \tau) = -\frac{i}{\eta^3(\tau)} \vartheta^{(1)} \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right] (z, \tau)$. In [Mason-Tuite-Z] the elliptic function version of the Fay's generalized trisecant identity [Fay] was derived. For $(\theta, \phi) \neq (1, 1)$ one has

$$\det(\mathbf{P}_n(\theta, \phi)) = \frac{\vartheta^{(1)} \left[\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix} \right] \left(\sum_{i=1}^n (x_i - y_i), \tau \right)}{\vartheta^{(1)} \left[\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix} \right] (0, \tau)} \cdot \frac{\prod_{1 \leq i < j \leq n} K^{(1)}(x_i - x_j, \tau) K^{(1)}(y_i - y_j, \tau)}{\prod_{1 \leq i, j \leq n} K^{(1)}(x_i - y_j, \tau)},$$

and similarly for $(\theta, \phi) = (1, 1)$,

$$\det(\mathbf{Q}_n) = -\frac{K^{(1)} \left(\sum_{i=1}^n (x_i - y_i), \tau \right) \prod_{1 \leq i < j \leq n} K^{(1)}(x_i - x_j, \tau) K^{(1)}(y_i - y_j, \tau)}{\prod_{1 \leq i, j \leq n} K^{(1)}(x_i - y_j, \tau)},$$

Higher power formulas

There exist also the analytic expansion [Mason-Tuite-Z], for $k, l \geq 1$,

$$P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z + z_1 - z_2, \tau) = \sum_{k, l \geq 1} D \begin{bmatrix} \theta \\ \phi \end{bmatrix} (k, l, z) z_1^{k-1} z_2^{l-1},$$
$$D \begin{bmatrix} \theta \\ \phi \end{bmatrix} (k, l, z, \tau) = (-1)^{k+1} \binom{k+l-2}{k-1} P_{k+l-1} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau, z).$$

Introduce is the block matrix

$$\mathbf{M}_{r,s} = \begin{pmatrix} \mathbf{D}^{(11)} & \dots & \mathbf{D}^{(1s)} \\ \vdots & \ddots & \vdots \\ \mathbf{D}^{(r1)} & \dots & \mathbf{D}^{(rs)} \end{pmatrix},$$

with $\mathbf{D}^{(ab)}$ the $m_a \times n_b$ matrix

$$\mathbf{D}^{(ab)}(i, j) = D \begin{bmatrix} \theta \\ \phi \end{bmatrix} (i, j, x_a - y_b, \tau), \quad (1 \leq i \leq m_a, 1 \leq j \leq n_b),$$

for $1 \leq a \leq r$ and $1 \leq b \leq s$.

Using the full version of the Fay's generalized trisecant identity [Mason-Tuite-Z], we derive the following

Proposition

For $(\theta, \phi) \neq (1, 1)$, $\zeta = 8\Phi$,

$$\Delta^\zeta(\tau) = (-i)^{\Phi/24} \frac{\vartheta^{(1)} \left[\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix} \right] (0, \tau) \Theta_{r,s,(m,n)}^{(1)}(x, y, \tau)}{\vartheta^{(1)} \left[\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix} \right] \left(\sum_{i=1}^r m_i x_i - \sum_{j=1}^s n_j y_j, \tau \right)} \det(\mathbf{M}_{r,s}),$$

where $\Phi = \sum_{1 \leq i \leq r, 1 \leq k \leq s} m_i n_j - \sum_{1 \leq i < k \leq r} m_i m_k - \sum_{1 \leq j < l \leq s} n_j n_l$.

Genus two formulas

In [Tuite-Z] we derived the genus two counterpart of the triple Jacobi identity. In particular, for $\alpha = \beta = 1/2$ one has

$$Z^{(1)} \begin{bmatrix} f_{1/2} \\ g_{1/2} \end{bmatrix} (\tau) = K^{(1)}(z, \tau) / \eta^2(\tau),$$

where $\zeta(x) = \sum_{i=1}^g \partial_{z_i} \vartheta \left[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix} \right] (0, \Omega) \nu_i(x)$, $K^{(1)}(x - y, \tau) = \frac{\vartheta^{(1)} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] \left(\int_y^x \nu, \tau \right)}{\zeta(x)^{\frac{1}{2}} \zeta(y)^{\frac{1}{2}}}.$

In [Tuite-Z] we compared the rank two fermion partition function on a genus two Riemann surface

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau_1, \tau_2, \epsilon) = Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (\tau_1) Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (\tau_2) \det \left(I - Q^{(1)} \right)^{1/2},$$

with it's the bosonized version

$$Z_M^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau_1, \tau_2, \epsilon) = \frac{1}{\eta(\tau_1) \eta(\tau_2) \det (I - A_1 A_2)^{1/2}} \vartheta^{(2)} \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left(\Omega^{(2)} \right).$$

Here

$$Q^{(1)} = \begin{pmatrix} 0 & \xi F_1^{(1)} \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} \\ -\xi F_2^{(1)} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} & 0 \end{pmatrix},$$

$$F_a^{(1)} \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, l, \tau_a, \epsilon) = (-1)^l \epsilon^{\frac{1}{2}(k+l-1)} \binom{k+l-2}{k-1} E_{k+l-1}^{(1)} \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (\tau_a),$$

$$A_a(k, l, \tau_a, \epsilon) = \epsilon^{(k+l)/2} \frac{(-1)^{k+1} (k+l-1)!}{\sqrt{kl} (k-1)! (l-1)!} E_{k+l}^{(1)} (\tau_a),$$

which gives

$$\begin{aligned} & \eta^3(\tau_1) \eta^3(\tau_2) \\ &= e^{2\pi i \alpha_{1/2} \cdot \beta_{1/2}} \frac{(K^{(1)}(z, \tau_1) K^{(1)}(z, \tau_2))^2}{\vartheta^{(2)} \begin{bmatrix} \alpha_{1/2} \\ \beta_{1/2} \end{bmatrix} (\Omega^{(2)})} \det(I - A_1 A_2)^{1/2} \det(I - Q^{(1)}) \end{aligned}$$

In particular, for $\tau = \tau_1 = \tau_2$ we obtain

$$\eta^6(\tau) = e^{2\pi i \alpha_{1/2} \cdot \beta_{1/2}} \frac{(K^{(1)}(z, \tau))^4}{\vartheta^{(2)} \begin{bmatrix} \alpha_{1/2} \\ \beta_{1/2} \end{bmatrix} (\Omega^{(2)})} \det(I - A_1 A_2)^{1/2} \det(I - Q^{(1)}).$$

The genus two: self-sewing formulas

In [Tuite-Z], by computing the genus two partition function for the fermionic vertex operator algebra and performing the bosonization, we found a genus two analogue of the classical Jacobi triple product identity

$$\frac{\vartheta^{(2)} \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\Omega^{(2)})}{\vartheta^{(1)} \left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \end{smallmatrix} \right] (\kappa w, \tau)} = e^{2i\pi\beta_2\kappa} \left(\frac{e^{i\pi B} \rho}{K^{(1)}(w, \tau)^2} \right)^{\frac{1}{2}\kappa^2} \det(I - T) \det(I - R)^{\frac{1}{2}}.$$

Here ρ is the torus self-sewing complex parameter, $-1/2 < \kappa < 1/2$, B is an odd integer parametrizing the formal branch cut, $T = \xi G D^{\theta_2}$,

$$R_{ab}(k, l) = -\frac{\rho^{(k+l)/2}}{\sqrt{kl}} \begin{bmatrix} D(k, l, \tau, w) & C(k, l, \tau) \\ C(k, l, \tau) & D(l, k, \tau, w) \end{bmatrix},$$

$D^{\theta_2}(k, l) = \begin{bmatrix} \theta_2^{-1} & 0 \\ 0 & -\theta_2 \end{bmatrix} \delta(k, l)$, $\xi \in \{\pm\sqrt{-1}\}$, and an infinite diagonal matrix:

$$G = \left[\frac{\rho^{\frac{1}{2}(k_a+l_b-1)}}{(2\pi i)^2} \oint_{C_{\bar{a}}(x_{\bar{a}})} \oint_{C_b(y_b)} (x_{\bar{a}})^{-k_a} (y_b)^{-l_b} S_{\kappa}(x_{\bar{a}}, y_b) dx_{\bar{a}}^{\frac{1}{2}} dy_b^{\frac{1}{2}} \right].$$

The genus two Szegő kernel for x, y taken on the torus is given by

$$\begin{aligned} S^{(2)}(x, y) &= S_{\kappa}^{(2)}(x, y) + \xi h(x) D^{\theta} (I - T)^{-1} \bar{h}^t(y) \\ &= \left[(x - y)^{-1} + \sum_{k, l} E_{k, l}^{(2)}(\Omega^{(2)}) x^{-k} y^l \right] dx^{\frac{1}{2}} dy^{\frac{1}{2}}, \end{aligned}$$

where $\bar{h}^t(y)$ denotes the transpose to

$$\bar{h}(y) = (\rho^{\frac{1}{2}(k_a - \frac{1}{2})} \frac{1}{2\pi i} \oint_{C_a(y_a)} y_a^{-k_a} S_{\kappa}^{(2)}(x, y_a) dy_a^{\frac{1}{2}}),$$

$$S_{\kappa}^{(2)}(x, y) = \left(\frac{\vartheta_1(x-w, \tau) \vartheta_1(y, \tau)}{\vartheta_1(x, \tau) \vartheta_1(y-w, \tau)} \right)^{\kappa} \frac{\vartheta \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (x-y+\kappa w, \tau)}{\vartheta \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (\kappa w, \tau) K(x-y, \tau)} dx^{\frac{1}{2}} dy^{\frac{1}{2}}.$$

Thus, we obtain [LSZ] a genus two formal generalization of the Garvan's formula:

$$\eta^{3\kappa^2}(\tau) = \frac{e^{-2i\pi\beta_2\kappa} \vartheta^{(2)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\Omega^{(2)}) \vartheta^{(1)} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (w, \tau)^{\kappa^2} \det \left[S^{(2)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (x, y) \right]}{(-e^{i\pi B} \rho)^{\frac{1}{2}\kappa^2} \vartheta^{(1)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (\kappa w, \tau) \det(I - R)^{\frac{1}{2}} \det \begin{bmatrix} S_{\kappa}^{(2)} & -\xi H D^{\theta_2} \\ \bar{H}^t & I - T \end{bmatrix}}$$

Looking forward for higher genus Eisenstein series

Thank you!

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