Uniqueness and multiplicity of large positive solutions

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Abstract: Establishing the uniqueness of the large positive solution of the semilinear equation

$$\Delta u = a(x)f(u) \tag{1}$$

on a bounded domain, Ω, of $\mathbb{R}^N$, $N \geq 1$, is imperative from the point of view of the applications of the theory of differential equations in Population Dynamics, [3]. When $a \equiv 1$, the existence of positive large solutions for (1) was established by J. B. Keller [1] and R. Osserman [10] in 1957. Essentially, in such case (1) has a large solution if and only if $f(u)$ satisfies the *Keller–Osserman condition*, which entails the existence of universal a priori bounds for the positive solutions of (1). Although there is a substantial amount of literature establishing the uniqueness of the large positive solution of (1) if $f(u)$ is increasing, among them, those of M. Marcus and L. Véron [9] and J. López-Gómez, L. Maire and M. Véron [7], as well as those of [2] and [4], even the precise role of the Keller–Osserman condition is far from well understood yet, [5].

This talk discusses some very recent advances on the problem of the uniqueness and multiplicity of the large positive solutions of (1). In particular, by adopting a dynamical perspective, constructs multiple large solutions of (1) when $f(u)$ is increasing, except on an arbitrarily small neighborhood of finitely many (prescribed) points where it exhibits a pulse-type behavior, and shows the uniqueness of the large solution if $f(u)$ is sufficiently close to an increasing function. Moreover, it discusses the recent multidimensional sharp uniqueness theorems of [7] and the astonishing one-dimensional multiplicity results of [8].

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