Semilinear elliptic problems involving Leray-Hardy potential and measure data

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Workshop: Singular Problems associated to Quasilinear Equations
In celebration of Marie Francoise Bidaut-Véron and Laurent Véron’s 70th birthday
Dear Prof. Bidaut-Véron and Prof. Véron, it is a great pleasure for me to participate in this wonderful meeting to celebrate such an important birthday.

I would like to take this opportunity to express my gratitude to you for your guidance and lots of assistance. I was most fortunate to be your and Prof. Felmer’s PhD student.
We will talk about

- elliptic equation with absorption nonlinearity and measure data, and elliptic equations with Hardy operators

- Isolated singular solutions of nonhomogeneous Hardy problem

\[ L_\mu u := -\Delta u + \frac{\mu}{|x|^2} u = f \quad \text{in} \quad \Omega \setminus \{0\}, \quad u = 0 \quad \text{on} \quad \partial \Omega \]

- semilinear Hardy equation involving measures

\[ L_\mu u + g(u) = \nu \quad \text{in} \quad \Omega \setminus \{0\}, \quad u = 0 \quad \text{on} \quad \partial \Omega \]

- solutions of nonhomogeneous Hardy problem with the origin on the boundary
Outline

1. Backgrounds
   - Laplacian operator
   - Hardy operator

2. Isolated singular solutions
   - Fundamental solution
   - Nonhomogeneous problem
   - Idea of proofs

3. Semilinear Hardy problem
   - Main results
   - The ideas of the proofs

4. Singular point on the boundary
• Benilan-Brezis-Crandall, Ann Sc Norm Sup Pisa (1975); Brezis, Appl Math Opim (1984)

For $p > 1$, $f \in L^1_{loc}(\mathbb{R}^N)$, the problem

$$\displaystyle - \Delta u + |u|^{p-1}u = f \quad \text{in} \quad \mathbb{R}^N \quad (1.1)$$

has a unique solution $u$. Moreover, $u \geq 0$ if $f \geq 0$.

• Lieb-Simon, Adv. Math (1977)

The Thomas-Fermi equation, Thomas-Fermi theory of atoms, molecules

$$\displaystyle - \Delta u + (u - \lambda)^{3/2} = \sum_{i=1}^{l} m_i \delta_{a_i} \quad \text{in} \quad \mathbb{R}^3, \quad (1.2)$$

where $\lambda \geq 0$, $m_i > 0$ and $\delta_{a_i}$ is the Dirac mass at $a_i \in \mathbb{R}^3$. The distributional solution of (1.2) is a classical solution of

$$\displaystyle -\Delta u + (u - \lambda)^{3/2} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \{a_1, \cdots, a_l\}. \quad (1.3)$$
A nature question is what difference between Dirac mass source and $L^1$ source.

- Benilan-Brezis, *J. Evol. Eq. (2004)* (finished 1975) answered this question, when $N \geq 3$, $p \geq \frac{N}{N-2}$, $k > 0$, the problem
  \[-\Delta u + |u|^{p-1}u = k\delta_0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \quad (1.4)\]
  has no solution.

- Brezis-Véron, *ARMA (1980)*: when $N \geq 3$, $p \geq N/(N-2)$, the basic model
  \[-\Delta u + |u|^{p-1}u = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \quad u = 0 \quad \text{on} \quad \partial\Omega \quad (1.5)\]
  admits only the zero nonnegative solution.
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admits only the zero nonnegative solution.
Veron, NA (1981)

For singularities of positive solutions of (1.5) for $1 < p < N/(N - 2)$ ($1 < p < \infty$ if $N = 2$), (when $(N + 1)/(N - 1) \leq p < N/(N - 2)$ the assumption of positivity is unnecessary) and that two types of singular behaviour occur:

- either $u(x) \sim c_N k |x|^{2-N}$ if $N \geq 3$, $u(x) \sim (-c_N k \ln |x|)$ if $N = 2$ as $|x| \to 0$ and $k$ can take any positive value; $u$ is said to have a **weak singularity** at $0$, and actually $u = u_k$, $u_k$ is a distributional solution of (1.4);

- or $u(x) \sim c_{N,p} |x|^{-\frac{2}{p-1}}$ as $x \to 0$; $u$ is said to have a **strong singularity** at $0$, and $u = u_\infty := \lim_{k \to \infty} u_k$. 
Chen-Matano-Veron, *JFA* (1989): *Anisotropic singularities*

When $1 < p < (N + 1)/(N - 1)$, $u$ is a solution of (1.5), then

- either $r^{2/p-1} u(r, \theta) \sim \omega(\theta)$, where $\omega$ is a solution of
  \[
  -\Delta_{S^{N-1}} \omega + |\omega|^{p-1} \omega = l_p \omega \quad \text{in} \quad S^{N-1};
  \]

- or there exists an integer $k < \frac{2}{p-1}$ and $\theta_0 \in [0, 2\pi)$ such that
  $u(r, \theta) \sim c_{N,q} kr^k \sin(k\theta + \theta_0)$ as $r = |x| \to 0$;

- or $u(x) \sim -c_N k \ln |x|$ as $|x| \to 0$.


For $N \geq 3$, the problem

\[
-\Delta u + g(u) = \nu \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega
\]  

(1.6)

has a unique distributional solution $u_\nu$ if $\nu$ is a bounded Radon measure, $g$ is nondecreasing locally Lipchitz continuous, $g(0) = 0$ and

\[
\int_{1}^{\infty} (g(s) - g(-s)) s^{-1 - \frac{N}{N-2}} ds < +\infty.
\]
When $N = 2$, introduced the exponential orders of growth of $g$ defined by

$$\beta_{\pm}(g) = \pm \inf \left\{ b > 0 : \int_1^\infty |g(\pm t)|e^{-bt} dt < \infty \right\}$$

(1.7)

if $\nu$ is any bounded measure in $\Omega$ with Lebesgue decomposition

$$\nu = \nu_r + \sum_{j \in \mathbb{N}} \alpha_j \delta_{a_j},$$

where $\nu_r$ is part of $\nu$ with no atom, $a_j \in \Omega$ and $\alpha_j \in \mathbb{R}$ satisfy

$$\frac{4\pi}{\beta_-(g)} \leq \alpha_j \leq \frac{4\pi}{\beta_+(g)},$$

(1.8)

then

$$- \Delta u + g(u) = \nu \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega$$

(1.9)

admits a unique weak solution.

When $g(u) = |u|^{p-1}u$ for $p > 1$ and they discovered that if $p \geq \frac{N}{N-2}$ the problem is well posed if and only if $\nu$ is absolutely continuous with respect to the Bessel capacity $c_{2,p'}$ with $p' = \frac{p}{p-1}$.
The Hardy inequalities

\[ \frac{(N - 2)^2}{4} \int_\Omega \frac{\xi^2}{|x|^2} \, dx \leq \int_\Omega |\nabla \xi|^2 \, dx, \quad \forall \xi \in H^1_0(\Omega); \]

Improved Hardy inequality

\[ \frac{(N - 2)^2}{4} \int_\Omega \frac{\xi^2}{|x|^2} \, dx + c \int_\Omega \xi^2 \, dx \leq \int_\Omega |\nabla \xi|^2 \, dx, \quad \forall \xi \in H^1_0(\Omega); \]

Denote

\[ \mu_0 = -\frac{(N - 2)^2}{4}. \]

Note that \( \mu_0 < 0 \) if \( N \geq 3 \) and \( \mu_0 = 0 \) if \( N = 2 \). Let Hardy operator be defined by

\[ \mathcal{L}_\mu = -\Delta + \frac{\mu}{|x|^2}. \]
Singular radial solutions of $L_\mu$

When $\mu \geq \mu_0$

$$L_\mu u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\} \quad (1.11)$$

has two branches of radial solutions with the explicit formulas that

$$\Phi_\mu(x) = \begin{cases} |x|^\tau_-(\mu) & \text{if} \quad \mu < \mu_0 \\ -|x|^\tau_-(\mu) \ln |x| & \text{if} \quad \mu = \mu_0 \end{cases} \quad \text{and} \quad \Gamma_\mu(x) = |x|^\tau_+(\mu), \quad (1.12)$$

where

$$\tau_-(\mu) = -\frac{N-2}{2} - \sqrt{\mu - \mu_0} \quad \text{and} \quad \tau_+(\mu) = -\frac{N-2}{2} + \sqrt{\mu - \mu_0}. \quad \text{Here the } \tau_-(\mu) \text{ and } \tau_+(\mu) \text{ are the zero points of } \tau(\tau + N - 2) - \mu = 0. \text{ In the following, we use the notations } \tau_- = \tau_-(\mu) \text{ and } \tau_+ = \tau_+(\mu).$$
semilinear Hardy problem

- the strong, $H^1_0$ and distributional solutions of

\[ \mathcal{L}_\mu u = u^p + tf, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \tag{1.13} \]

- a classical solution $u$ is a $C^2(\bar{\Omega} \setminus \{0\})$ function verifies the equation pointwise in $\Omega \setminus \{0\}$ and $u(x) \leq c \Gamma_\mu$ for some $c > 0$;
- a $H^1$ solution $u$ is a $H^1_0(\Omega)$ function verifies the identity

\[ \int_\Omega (\nabla u \nabla \xi - \frac{\mu}{|x|^2} u \xi) = \int_\Omega (u^p + tf) \xi, \quad \forall \xi \in H^1_0(\Omega); \]

- a distributional solution $u$, if $u \in L^1(\Omega)$, $\frac{u}{|x|^2} \in L^1(\Omega, \rho dx)$ and $u$ verifies that

\[ \int_\Omega u \mathcal{L}_\mu \xi = \int_\Omega (u^p + tf) \xi, \quad \forall \xi \in C^2(\bar{\Omega}) \cap C_0(\Omega), \]

where $\rho(x) = \text{dist}(x, \partial \Omega)$. 
Dupaigne’s main results

**Theorem**

Assume that $N \geq 3$, $\mu \in [\mu_0, 0)$, $f$ is a smooth, bounded and nonnegative function and

$$q^*_\mu = 1 + \frac{2}{-\tau_+(\mu)}$$

For $1 < p < q^*_\mu$, there exists $t_0$ such that

(i) if $0 < t < t_0$, problem (1.13) has a minimal classical solution;
(ii) if $t = t_0$, problem (1.13) has a minimal distributional solution;
(iii) if $t > t_0$, problem (1.13) has no distributional solution.

* Brezis-Dupaigne-Tesei *Sel Math* (2005)

When $t = 0$, (1.13) has a nontrivial nonnegative solution if $p < q^*_\mu$ and does not have nonnegative distributional solutions for $p \geq q^*_\mu$. 
Guerch and Véron, *Rev mat Iberoamericana* 1991

- $\mu > \mu_0$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(0) \geq 0$

\[
\int_1^\infty (g(s) - g(-s))s^{-1 - \frac{r-2}{r}} \, ds < \infty; \quad (1.14)
\]

- $\mu = \mu_0$, $k > 0$, $N \geq 3$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(0) \geq 0$ and

\[
\int_1^\infty g \left( kt \frac{N-2}{N+2} \ln t \right) t^{-2} \, dt < \infty, \quad (1.15)
\]

semilinear Hardy problem

\[
\mathcal{L}_\mu u + g(u) = 0 \quad \text{in } \Omega \setminus \{0\}, \quad u = 0 \quad \text{on } \partial \Omega \quad (1.16)
\]

has a classical solution $u_k \in C^2(\bar{\Omega} \setminus \{0\})$ such that $\lim_{|x| \to 0} \frac{u_k(x)}{\Phi_\mu(x)} = k$. 
Cîrstea, *American mathematical society 2014*

The positive solution of semilinear Hardy equation $\mathcal{L}_\mu u + g(u) = 0$ in $\Omega \setminus \{0\}$ has three possible singularities at the origin:

\[
\text{either } \lim_{x \to 0} \frac{u(x)}{\Phi_\mu(x)} = +\infty \text{ or } \lim_{x \to 0} \frac{u(x)}{\Phi_\mu(x)} \in (0, +\infty),
\]

\[
\text{or } \lim_{x \to 0} \frac{u(x)}{\Gamma_\mu(x)} \in (0, +\infty).
\]

Related elliptic problem with boundary Hardy potential:

- Gkikas-Véron, *NA 2015*
- Nguyen, *CVPDE 2017;*
Some questions

- When $\mu = 0$, $\Phi_0(x) = |x|^{2-N}$ if $N \geq 3$ and $\Gamma_\mu = 1$, function $\Phi_0$ verifies the distributional identity

$$\int_{\mathbb{R}^N} \Phi_0 \mathcal{L}_0 \xi \, dx = c_0 \xi(0), \quad \forall \xi \in C^2_c(\mathbb{R}^N)$$

- For $\mu \in [\mu_0, 0)$, there holds that

$$\int_{\mathbb{R}^N} \Phi_\mu \mathcal{L}_\mu \xi \, dx = \int_{\mathbb{R}^N} \Gamma_\mu \mathcal{L}_\mu \xi \, dx = 0, \quad \forall \xi \in C^2_c(\mathbb{R}^N) \quad (1.19)$$

For $\mu \in [\mu_0, 0)$, the Dirac mass can not be used to express the singularities of the function $\Phi_\mu$ or $\Gamma_\mu$ in the traditional distributional sense.

- Especially, when $\mu > 0$ large enough, the distributional identity (1.19) for $\Phi_\mu$ is not well-defined.
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2 Isolated singular solutions
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3 semilinear Hardy problem
   - Main results
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4 Singular point on the boundary
When \( \mu \geq \mu_0 \), \( \Phi_\mu \) and \( \Gamma_\mu \) satisfy \( \mathcal{L}_\mu u = 0 \) in \( \mathbb{R}^N \setminus \{0\} \).

**Theorem**

Let \( d\gamma_\mu(x) = \Gamma_\mu(x)dx \) and

\[
\mathcal{L}^*_\mu = -\Delta - 2\frac{\tau_+(\mu)}{|x|^2} x \cdot \nabla. \tag{2.1}
\]

Then

\[
\int_{\mathbb{R}^N} \Phi_\mu \mathcal{L}^*_\mu(\xi) d\gamma_\mu = c_\mu \xi(0), \quad \forall \xi \in C^2_c(\mathbb{R}^N), \tag{2.2}
\]

where

\[
c_\mu = \begin{cases} 
2\sqrt{\mu - \mu_0} |\mathbb{S}^{N-1}| & \text{if } \mu > \mu_0, \\
|\mathbb{S}^{N-1}| & \text{if } \mu = \mu_0.
\end{cases} \tag{2.3}
\]

In fact we show that

$$\Gamma_{\mu} \cdot \mathcal{L}_{\mu}(\Phi_{\mu}) = c_{\mu} \delta_0.$$  \hspace{1cm} (2.4)

In particular, for $\mu = 0$, $\Gamma_{\mu} = 1$, $\mathcal{L}_{\mu}^* = -\Delta$ and (2.4) reduces to

$$-\Delta \Phi_0 = c_0 \delta_0.$$

Observation: $\tau_-(\mu) + \tau_+(\mu) = 2 - N$, for $\xi \in C^2_c(\mathbb{R}^N)$, we use test function $\Gamma_{\mu} \xi$,

$$0 = \int_{\mathbb{R}^N \setminus B_r(0)} \mathcal{L}_{\mu}(\Phi_{\mu}) \Gamma_{\mu} \xi \, dx$$

$$= \int_{\mathbb{R}^N \setminus B_r(0)} \Phi_{\mu} \mathcal{L}_{\mu}^*(\xi) \, d\gamma_{\mu} + \int_{\partial B_r(0)} \left( \nabla \Phi_{\mu} \cdot \frac{x}{|x|} \Gamma_{\mu} - \nabla \Gamma_{\mu} \cdot \frac{x}{|x|} \Phi_{\mu} \right) \xi \, d\omega$$

$$- \int_{\partial B_r(0)} \Phi_{\mu} \Gamma_{\mu} \left( \nabla \xi \cdot \frac{x}{|x|} \right) \, d\omega.$$

Here $\Phi_{\mu}$ is said to be a fundamental solution of $\mathcal{L}_{\mu}$. We note that the fundamental solution $\Phi_{\mu}$ keeps positive when $\mu < \mu_0$ and changes signs for $\mu = \mu_0$.
In the bounded $C^2$ domain $\Omega$ containing the origin,

$$\begin{cases} 
\mathcal{L}_\mu u = 0 & \text{in } \Omega \setminus \{0\}, \\
u = 0 & \text{on } \partial \Omega, \\
\lim_{x \to 0} u(x) \Phi^{-1}_\mu(x) = 1 
\end{cases} \quad (2.5)$$

has a unique solution $\Phi_{\mu, \Omega}$.

**Theorem**

Let $\Phi_{\mu, \Omega}$ be the solution of (2.5), then

$$\int_{\Omega} \Phi_{\mu, \Omega} \mathcal{L}^*_\mu(\xi) \, d\gamma = c_\mu \xi(0), \quad \forall \xi \in C^{1,1}_0(\Omega). \quad (2.6)$$
Approximation of the fundamental solution

Let \( \{\delta_n\}_n \) be a sequence of nonnegative \( L^\infty \) functions that
\[ \text{supp} \delta_n \subset B_{r_n}(0), \text{where} \ r_n \to 0 \text{ as} \ n \to +\infty, \]
\[ \delta_n \to \delta_0 \text{ as} \ n \to +\infty \text{ in the distributional sense}. \]

For any \( n \), the problem
\[
\begin{cases}
    \mathcal{L}_\mu u = c_\mu \delta_n / \Gamma_\mu & \text{in} \ \Omega \setminus \{0\}, \\
    u = 0 & \text{on} \ \partial\Omega, \\
    \lim_{x \to 0} u(x)\Phi^{-1}_\mu(x) = 0
\end{cases}
\]

has a unique solution \( w_n \).

Then
\[
\lim_{n \to +\infty} w_n(x) = \Phi_{\mu,\Omega}(x), \ \forall x \in \Omega \setminus \{0\}.
\]
We consider nonhomogeneous problem

$$\mathcal{L}_\mu u = f \text{ in } \Omega \setminus \{0\}, \quad u = 0 \text{ on } \partial \Omega. \quad (2.8)$$

**Theorem**

Let $\mu \geq \mu_0$, $f$ be a function in $C^\theta_{loc}(\overline{\Omega} \setminus \{0\})$ for some $\theta \in (0, 1)$.

1. **(i)** Assume that

$$\int_{\Omega} |f| \, d\gamma_\mu < +\infty, \quad (2.9)$$

then problem (2.8), subject to $\lim_{x \to 0} u(x) \Phi^{-1}_\mu(x) = k$ with $k \in \mathbb{R}$, has a unique solution $u_k$, which satisfies the distributional identity

$$\int_{\Omega} u_k \mathcal{L}^*_\mu(\xi) \, d\gamma_\mu = \int_{\Omega} f\xi \, d\gamma_\mu + c_\mu k \xi(0), \quad \forall \xi \in C^{1,1}_0(\Omega). \quad (2.10)$$

2. **(ii)** Assume that $f$ verifies (2.9) and $u$ is a nonnegative solution of (2.8), then $u$ satisfies (2.10) for some $k \geq 0$.

3. **(iii)** Assume that $f \geq 0$ and

$$\lim_{r \to 0^+} \int_{\Omega \setminus B_r(0)} f \, d\gamma_\mu = +\infty, \quad (2.11)$$

then problem (2.8) has no nonnegative solutions.
Part 1: existence for \( f \in L^1(\Omega, d\gamma_{\mu}) \)

**Lemma**

Assume that \( f \in C^\theta(\bar{\Omega}) \) for some \( \theta \in (0, 1) \), then

\[
\begin{cases}
\mathcal{L}_{\mu} u = f & \text{in } \Omega \setminus \{0\}, \\
u = 0 & \text{on } \partial\Omega, \\
\lim_{x \to 0} u(x)\Phi_{\mu}^{-1}(x) = 0
\end{cases}
\]  

(2.12)

has a unique solution \( u_f \) satisfying the distributional identity:

\[
\int_\Omega u_f \mathcal{L}_{\mu}^*\xi \, d\gamma_{\mu} = \int_\Omega f\xi \, d\gamma_{\mu}, \quad \forall \xi \in C^{1,1}_0(\Omega).
\]  

(2.13)

- **The case** \( \mu > \mu_0 \). Indeed, for \( \mu > \mu_0 \), we can choose \( \tau_0 \in (\tau-(\mu), \min\{2, \tau+(\mu)\}) \), and denote

\[
V_0(x) = |x|^{\tau_0}, \quad \forall \, x \in \Omega \setminus \{0\}.
\]

Then

\[
\mathcal{L}_{\mu} V_0(x) = c_{\tau_0} |x|^{\tau_0 - 2},
\]

where \( c_{\tau_0} = \mu - \tau_0(\tau_0 + N - 2) > 0 \).
Since $f$ is bounded, there exists $t_0 > 0$ such that

$$|f(x)| \leq t_0 c \tau_0 |x|^{\tau_0 - 2}, \quad \forall x \in \Omega \setminus \{0\},$$

then $t_0 V_0$ and $-t_0 V_0$ are supersolution and subsolution of (2.12) respectively.

- **The case $\mu = \mu_0$ and $N \geq 3$.**
  - $\mu \mapsto u_\mu$ is decreasing in $[\mu_0, 0)$.
  - A uniformly bound for $u_\mu$ for $\mu > \mu_0$

$$V(x) = |x|^\tau_+ (\mu_0) - (s_0 |x|)^2, \quad \forall x \in \Omega \setminus \{0\},$$

where $s_0 > 0$ and $V > 0$ in $\Omega \setminus \{0\}$. Then there exists $t_0 > 0$ such that

$$u_\mu \leq t_0 V \quad \text{in} \quad \Omega \setminus \{0\}.$$

For $\xi \in C^{1.1}_0(\Omega)$, there exists $c > 0$ independent of $\mu$ such that

$$|\mathcal{L}_\mu^*(\xi)| \leq c \|\xi\|_{C^{1.1}_0(\Omega)} + |\mu| \|\xi\|_{C^1(\Omega)} |x|^{-1}.$$

- From the dominate monotonicity convergence theorem, there exists $u_{\mu_0} \leq t_0 V$ such that

$$u_\mu \to u_{\mu_0} \quad \text{as} \quad \mu \to \mu_0^+ \quad \text{a.e. in} \quad \Omega \quad \text{and in} \quad L^1(\Omega, |x|^{-1} d\gamma_\mu)$$

and

$$\int_{\Omega} u_{\mu_0} \mathcal{L}_{\mu_0}^*(\xi) d\gamma_{\mu_0} = \int_{\Omega} f(x) d\gamma_{\mu_0}$$
Part 2: nonexistence for $f \not\in L^1(\Omega, d\gamma_\mu)$

- From (2.11) and the fact $f \in C^\theta(\Omega \setminus \{0\})$, for any $r_n$, we have that

$$\lim_{r \to 0^+} \int_{B_{r_n}(0) \setminus B_r(0)} f(x) d\gamma_\mu = +\infty,$$

then there exists $R_n \in (0, r_n)$ such that $\int_{B_{r_n}(0) \setminus B_{R_n}(0)} f d\gamma_\mu = n$.

Let $\delta_n = \frac{1}{n} \Gamma_\mu f \chi_{B_{r_n}(0) \setminus B_{R_n}(0)}$, then the problem

$$\begin{cases}
L_\mu u \cdot \Gamma_\mu = \delta_n & \text{in } \Omega \setminus \{0\}, \\
u = 0 & \text{on } \partial \Omega, \\
\lim_{x \to 0} u(x) \Phi_\mu^{-1}(x) = 0
\end{cases}$$

has a unique positive solution $w_n$ satisfying

$$\int_{\Omega} w_n L_\mu (\Gamma_\mu \xi) dx = \int_{\Omega} \delta_n \xi dx, \quad \forall \xi \in C_0^{1,1}(\Omega).$$
For any $\xi \in C^{1,1}_0(\Omega)$, we have that
\[
\int_{\Omega} w_n \mathcal{L}_\mu^*(\xi) \, d\gamma_\mu = \int_{\Omega} \delta_n \xi \, dx \to \xi(0) \quad \text{as} \quad n \to +\infty.
\]
Therefore for any compact set $K \subset \Omega \setminus \{0\}$,
\[
\|w_n - \Phi_{\mu,\Omega}\|_{C^1(K)} \to 0 \quad \text{as} \quad n \to +\infty.
\]
Fix $x_0 \in \Omega \setminus \{0\}$ and $r_0 = \frac{\min\{|x_0|, \rho(x_0)|}{2}$ and $K = \overline{B_{r_0}(x_0)}$, then there exists $n_0 > 0$ such that for $n \geq n_0$,
\[
w_n \geq \frac{1}{2} \Phi_{\mu,\Omega} \quad \text{in} \quad K. \tag{2.14}
\]
Let $u_n$ be the solution of
\[
\begin{cases}
\mathcal{L}_\mu u \cdot \Gamma_\mu = n \delta_n \quad \text{in} \quad \Omega \setminus \{0\}, \\
u = 0 \quad \text{on} \quad \partial \Omega, \\
\lim_{x \to 0} u(x) \Phi_\mu^{-1}(x) = 0,
\end{cases}
\]
thus, together with (2.14), we have that
\[
u_n \geq n w_n \geq \frac{n}{2} \Phi_{\mu,\Omega} \quad \text{in} \quad K
\]
and
\[
u_f(x_0) \geq u_n(x_0) \to +\infty \quad \text{as} \quad n \to +\infty,
\]
which contradicts that $u_f$ is classical solution of (2.8).
Part 3: nonexistence when $\mu < \mu_0$

**Theorem**

Assume that $\mu < \mu_0$ and $f$ is a measurable nonnegative function, then problem (2.8) has no nontrivial nonnegative solutions.

**Sketch of the proof.** Let $u_0$ be a nontrivial nonnegative solution of (2.8).

\[ \mathcal{L}_{\mu_0} u_0 = (\mu_0 - \mu) \frac{u_0}{|x|^2} + f \geq (\mu_0 - \mu) \varepsilon_0 \frac{\chi_{B_{r_0}(x_0)}}{|x|^2}, \]

When $N \geq 3$, for $x \in B_{r_0}(0) \setminus \{0\}$,

\[ u_0(x) \geq (\mu_0 - \mu) \varepsilon_0 \mathcal{G}_{\mu_0}[\chi_{B_{r_0}(x_0)}] \geq c_0 |x|^{-\frac{N-2}{2}}, \]

then

\[ \int_{\Omega \setminus B_r(0)} \left[ (\mu_0 - \mu) \frac{u_0}{|x|^2} + f \right] d\gamma_{\mu_0} \geq c_0 \int_{B_{r_0}(0) \setminus B_r(0)} |x|^{-N} dx \]

\[ \rightarrow +\infty \quad \text{as} \quad r \rightarrow 0^+. \]

We obtain that

\[ \mathcal{L}_\mu u = (\mu_0 - \mu) \frac{u_0}{|x|^2} + f \quad \text{in} \quad \Omega \setminus \{0\}, \quad u = 0 \quad \text{on} \quad \partial \Omega \quad (2.15) \]

has no nonnegative solution.
## Outline

1. **Backgrounds**
   - Laplacian operator
   - Hardy operator

2. **Isolated singular solutions**
   - Fundamental solution
   - Nonhomogeneous problem
   - Idea of proofs

3. **Semilinear Hardy problem**
   - Main results
   - The ideas of the proofs

4. **Singular point on the boundary**
The nonlinear Poisson equation

\[ \mathcal{L}_\mu u + g(u) = \nu \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega, \quad (3.1) \]

where \( \mu \geq \mu_0, \ g : \mathbb{R} \to \mathbb{R} \) is a continuous nondecreasing function such that \( g(0) \geq 0 \) and \( \nu \) is a Radon measure in \( \Omega \).

- we denote by \( \mathcal{M}(\Omega^*; \Gamma_\mu) \), the set of Radon measures \( \nu \) in \( \Omega^* \) such that

\[
\int_{\Omega^*} \Gamma_\mu d|\nu| := \sup \left\{ \int_{\Omega^*} \zeta d|\nu| : \zeta \in C_c(\Omega^*), \ 0 \leq \zeta \leq \Gamma_\mu \right\} < \infty, \quad (3.2)
\]

where \( \Omega^* = \Omega \setminus \{0\} \).

- we denote by \( \mathcal{M}(\Omega; \Gamma_\mu) \) the set of measures \( \nu \) on \( \Omega \) which coincide with the above natural extension of \( \nu|_{\Omega^*} \in \mathcal{M}_+(\Omega^*; \Gamma_\mu) \). If \( \nu \in \mathcal{M}_+(\Omega; \Gamma_\mu) \) we define the measure \( \Gamma_\mu \nu \) in the following way

\[
\int_{\Omega} \zeta d(\Gamma_\mu \nu) = \sup \left\{ \int_{\Omega^*} \eta \Gamma_\mu d\nu : \eta \in C_c(\Omega^*), \ 0 \leq \eta \leq \zeta \right\} \quad \text{for all} \ \zeta \in C_c(\Omega), \ \zeta \geq 0. \quad (3.3)
\]
• We denote by $\mathcal{M}(\Omega; \Gamma_\mu)$ the set of measures which can be written under the form

$$\nu = \nu|_{\Omega^*} + k\delta_0,$$

(3.4)

where $\nu|_{\Omega^*} \in \mathcal{M}(\Omega; \Gamma_\mu)$ and $k \in \mathbb{R}$.

• We denote $\overline{\Omega}^* := \overline{\Omega} \setminus \{0\}$, $\rho(x) = \text{dist}(x, \partial \Omega)$ and

$$\mathbb{X}_\mu(\Omega) = \left\{ \xi \in C_0(\overline{\Omega}) \cap C^1(\overline{\Omega}^*) : |x| L^\ast \xi \in L^\infty(\Omega) \right\}.$$  

(3.5)

Clearly, $C^1_{0,1}(\overline{\Omega}) \subset \mathbb{X}_\mu(\Omega)$.

**Definition**

• We say that $u$ is a weak solution of (3.1) with $\nu \in \mathcal{M}(\Omega; \Gamma_\mu)$ such that $\nu = \nu|_{\Omega^*} + k\delta_0$ if $u \in L^1(\Omega, |x|^{-1} d\gamma_\mu)$, $g(u) \in L^1(\Omega, \rho d\gamma_\mu)$ and

$$\int_{\Omega} \left[ u L^\ast \mu + g(u) \xi \right] d\gamma_\mu = \int_{\Omega} \xi d(\Gamma_\mu \nu) + c_\mu k \xi(0) \quad \text{for all } \xi \in \mathbb{X}_\mu(\Omega).$$

(3.6)
- the Dirac mass at 0 does not belong to $\mathcal{M}(\Omega; \Gamma_\mu)$ although it is a limit of 
  \( \{\nu_n\} \subset \mathcal{M}(\Omega; \Gamma_\mu) \).

**Definition**

- A continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $rg(r) \geq 0$ for all $r \in \mathbb{R}$ satisfies the weak $\Delta_2$-condition if there exists a positive nondecreasing function $t \mapsto K(t)$ such that

  $$\left| g(s + t) \right| \leq K(t) \left( |g(s)| + |g(t)| \right) \quad \text{for all } (s, t) \in \mathbb{R} \times \mathbb{R} \text{ s.t. } st \geq 0. \quad (3.7)$$

It satisfies the $\Delta_2$-condition if the above function $K$ is constant.

- Critical exponent

  $$p^*_\mu = 1 - \frac{2}{\tau_-}. \quad (3.8)$$

Note that $p^*_\mu < p^*_0$ if $\mu > 0$ and $p^*_\mu > p^*_0$ if $\mu < 0$. 

**Theorem**

Let \( \mu > 0 \) if \( N = 2 \), \( \mu \geq \mu_0 \) if \( N \geq 3 \) and \( g : \mathbb{R} \to \mathbb{R} \) be a Hölder continuous nondecreasing function such that \( g(0) = 0 \). Then for any \( \nu \in L^1(\Omega, d\gamma_\mu) \), problem (3.1) has a unique weak solution \( u_\nu \) such that for some \( c_1 > 0 \),

\[
\|u_\nu\|_{L^1(\Omega, |x|^{-1}d\gamma_\mu)} \leq c_1 \|\nu\|_{L^1(\Omega, d\gamma_\mu)}.
\]

Furthermore, if \( u_{\nu'} \) is the solution of (3.1) with right-hand side \( \nu' \in L^1(\Omega, d\gamma_\mu) \), there holds

\[
\int_\Omega \left[ |u_\nu L^*_\mu \xi + g(u_\nu))| \xi \right] d\gamma_\mu \leq \int_\Omega (\nu) \text{sgn}(u_\nu) \xi d\gamma_\mu \tag{3.9}
\]

and

\[
\int_\Omega \left[ (u_\nu + L^*_\mu \xi + (g(u_\nu)) + \xi \right] d\gamma_\mu \leq \int_\Omega \nu \text{sgn}+(u_\nu) \xi d\gamma_\mu \tag{3.10}
\]

for all \( \xi \in X_\mu(\Omega), \xi \geq 0 \), where \( \text{sgn}(t) = 1 \) if \( t > 0 \), \( \text{sgn}(0) = 0 \) and \( \text{sgn}(t) = -1 \) if \( t < 0 \).

• Remark: (3.9) and (3.10) are Kato’s type Inequalities; these inequalities plays an important role in the derivation of uniqueness.
Now we state the existence of weak solution in the subcritical case with $\mu > \mu_0$.

**Theorem**

Let $\mu > \mu_0$ and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $g(r)r \geq 0$ for any $r \in \mathbb{R}$. If $g$ satisfies the weak $\Delta_2$-condition and

$$
\int_1^\infty (g(s) - g(-s)) s^{1 - \min\{p_\ast^\mu, p_\ast^0\}} ds < \infty. \tag{3.11}
$$

Then for $\nu \in \overline{\mathcal{M}}_+(\Omega; \Gamma_\mu)$ problem (3.1) admits a unique weak solution $u_\nu$. Furthermore, the mapping: $\nu \mapsto u_\nu$ is increasing.

- For $\nu = \nu|_{\Omega^*} + c_\mu k \delta_0 \in \overline{\mathcal{M}}_+(\Omega; \Gamma_\mu)$ and $g(t) = |t|^{p-1}t$, problem (3.1) has a unique solution if
  
  (i) $1 < p < p_\ast^\mu$ in the case $\nu|_{\Omega^*} = 0$;
  
  (ii) $1 < p < p_\ast^0$ in the case $k = 0$;
  
  (iii) $1 < p < \min\{p_\ast^\mu, p_\ast^0\}$ in the case $k \neq 0$ and $\nu|_{\Omega^*} \neq 0$.

- Examples: Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$ and $\nu = \sum_{n=1}^\infty a_n \delta e_1 + k \delta_0$, where $a_n > 0$ is such that $\sum_{n=1}^\infty a_n^{\tau^+} < +\infty$. 

Theorem

Assume that $N \geq 3$, $\mu = \mu_0$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(r)r \geq 0$ for any $r \in \mathbb{R}$ satisfying the weak $\Delta_2$-condition and

$$\int_1^{+\infty} (g(s) - g(-s))s^{-1-N\frac{2}{N-2}} ds < +\infty. \quad (3.12)$$

Then for any $\nu = \nu|_{\Omega^*} + c_\mu k\delta_0 \in \overline{M}_+(\Omega; \Gamma_\mu)$ problem (3.1) admits a unique weak solution $u_\nu$.

Furthermore, if $\nu|_{\Omega^*} = 0$, condition (3.12) can be replaced by the following weaker one

$$\int_1^{+\infty} (g(t) - g(-t))(\ln t)^{\frac{N+2}{N-2}} t^{-1-N\frac{2}{N-2}} dt < \infty. \quad (3.13)$$

- Examples: $\nu = k\delta_0$ and $g(t) = t^{\frac{N+4}{N-2}}(\ln t)^{\tau}$ with $\tau > 2N\frac{N-2}{N}$, (3.1) has an isolated singular solution $u_{k} > 0$. 
In the supercritical case, we set $g_p(u) = |u|^{p-1} u$, i.e.

$$\mathcal{L}_\mu u + g_p(u) = \nu \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,$$

(3.14)

**Theorem**

Assume that $N \geq 3$. Then $\nu = \nu|_{\Omega^*} \in \mathcal{M}(\Omega; \Gamma_\mu)$ is $g_p$-good if and only if for any $\epsilon > 0$, $\nu_\epsilon = \nu \chi_{B_{\epsilon}}$ is absolutely continuous with respect to the $c_{2,p'}$-Bessel capacity.

Finally we characterize the compacts removable sets in $\Omega$.

**Theorem**

Assume that $N \geq 3$, $p > 1$ and $K$ is a compact set of $\Omega$. Then any weak solution of

$$\mathcal{L}_\mu u + g_p(u) = 0 \text{ in } \Omega \setminus K$$

(3.15)

can be extended a solution of the same equation in whole $\Omega$ if and only if

(i) $c_{2,p'}(K) = 0$ if $0 \notin K$;

(ii) $p \geq p_\mu^*$ if $K = \{0\}$;

(iii) $c_{2,p'}(K) = 0$ if $\mu \geq 0$, $0 \in K \text{ and } K \setminus \{0\} \neq \{0\}$;

(iv) $c_{2,p'}(K) = 0$ and $p \geq p_\mu^*$ if $\mu < 0$, $0 \in K \text{ and } K \setminus \{0\} \neq \{0\}$. 
Part 1: linear problem

Lemma

If \( \nu \in \mathcal{M}(\Omega; \Gamma_\mu) \), then

\[
\mathcal{L}_\mu u = \nu \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. 
\]

(3.16)

admits a unique solution in \( L^1(\Omega, |x|^{-1}d\gamma_\mu) \), denoted by \( G_\mu[\nu] \), and this defines the Green operator of \( \mathcal{L}_\mu \) in \( \Omega \) with homogeneous Dirichlet conditions.

- Let \( \{\nu_n\} \subset L^1(\Omega, \rho d\gamma_\mu) \) be a sequence such that \( \nu_n \geq 0 \) and

\[
\int_\Omega \xi \Gamma_\mu \nu_n dx \rightarrow \int_\Omega \xi d(\Gamma_\mu \nu) \quad \text{for all} \quad \xi \in X_\mu(\Omega),
\]

with \( n \in \mathbb{N} \), the weak solution of

\[
\mathcal{L}_\mu u_n = \nu_n \quad \text{in} \quad \Omega, \quad u_n = 0 \quad \text{on} \quad \partial\Omega 
\]

(3.17)

satisfies that for any open sets \( O \) verifying \( \bar{O}_\Omega \setminus B_\epsilon(0) \) for some \( c > 0 \) independent of \( n \) but dependent of \( O' \),

\[
\|u_n\|_{W^{1,q}(O)} \leq c \|\nu\|_{\mathcal{M}(\Omega, \Gamma_\mu)}.
\]

That is, \( \{u_n\} \) is uniformly bounded in \( W^{1,q}_{loc}(\Omega \setminus \{0\}) \).
The ideas of the proofs

- Let $\omega \subset \Omega$ be a Borel set and the solution $\psi_\omega$ of

$$
\begin{cases}
  \mathcal{L}_\mu^* \psi_\omega = |x|^{-1} \chi_\omega & \text{in } \Omega, \\
  \psi_\omega = 0 & \text{on } \partial \Omega
\end{cases}
$$

has the property

$$
\lim_{|\omega| \to 0} \psi_\omega(x) = 0 \text{ uniformly in } B_1
$$

and

$$
\int_{\omega} \frac{u_n}{|x|} d\gamma_\mu(x) = \int_{\omega} \nu_n \Gamma_\mu \psi_\omega dx \leq \sup_{\Omega} \psi_\omega \int_{\omega} \nu_n \Gamma_\mu dx \to 0 \text{ as } |\omega| \to 0.
$$

This shows that $\{u_n\}$ is uniformly integrable for the measure $|x|^{-1} d\gamma_\mu$. 
Part 2: Isolated singular solutions

Lemma

Let \( k \in \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous nondecreasing function such that \( rg(r) \geq 0 \) for all \( r \in \mathbb{R} \). Then problem

\[
\begin{align*}
\mathcal{L}_\mu u + g(u) &= k\delta_0 \quad \text{in} \; \Omega, \\
 u &= 0 \quad \text{on} \; \partial\Omega
\end{align*}
\]

admits a unique solution \( u := u_{k\delta_0} \) if one of the following conditions is satisfied:

(i) \( N = 2, \mu > \mu_0 \) and \( g \) satisfies

\[
\int_1^\infty (g(s) - g(-s)) s^{-1-p^*_\mu} ds < \infty;
\]

(ii) \( N \geq 3, \mu = \mu_0 \) and \( g \) satisfies (3.13).

For \( \mu > \mu_0 \) [Guerch-Veron 1991] for any \( k \in \mathbb{R} \) there exists a radial function \( v_{k,1} \) (resp. \( v_{k,R} \)) defined in \( B^*_1 \) (resp. \( B^*_R \)) satisfying

\[
\mathcal{L}_\mu v + g(v) = 0 \quad \text{in} \; B^*_1 \quad \text{(resp. in} \; B^*_R),
\]

vanishing respectively on \( \partial B_1 \) and \( \partial B_R \) and satisfying

\[
\lim_{x \to 0} \frac{v_{k,1}(x)}{\Phi_\mu(x)} = \lim_{x \to 0} \frac{v_{k,R}(x)}{\Phi_\mu(x)} = \frac{k}{c_\mu}.
\]
For $\mu = \mu_0$, [Guerch-Veron 1991] shows the existence of isolated singular solution if for some $b > 0$ there holds

$$I := \int_1^\infty g \left( b t^{\frac{N-2}{N+2}} \ln t \right) t^{-2} dt < \infty,$$

(3.23)

set $s = t^{\frac{N-2}{N+2}}$ and $\beta = \frac{N+2}{N-2} b$, then

$$I = \frac{N + 2}{N - 2} \int_1^\infty g (\beta s \ln s) s^{-\frac{2N}{N-2}} ds$$

Set $\tau = \beta s \ln s$, then

$$\ln \tau = \ln s \left( 1 + \frac{\ln \ln s}{\ln s} + \frac{\ln \beta}{\ln s} \right) \implies \ln s = \ln \tau (1 + o(1)) \text{ as } s \to \infty.$$  

We infer that for $\epsilon > 0$ there exists $s_\epsilon > 2$ and $\tau_\epsilon = s_\epsilon \ln s_\epsilon$ such that

$$(1 - \epsilon) \beta^{\frac{N+2}{N-2}} \leq \frac{\int_{s_\epsilon}^\infty g (\beta s \ln s) s^{-\frac{2N}{N-2}} ds}{\int_{\tau_\epsilon}^\infty g (\tau) (\ln \tau)^{\frac{N+2}{N-2}} \tau^{-\frac{2N}{N-2}} d\tau} \leq (1 + \epsilon) \beta^{\frac{N+2}{N-2}}.$$  

(3.24)

Thus, $I < +\infty$ is equivalent to (3.13).
Part 3: Measures in $\Omega^*$

$$L_\mu u + g(u) = \nu \quad \text{in} \; \Omega, \quad u = 0 \quad \text{on} \; \partial \Omega \quad (3.25)$$

**Lemma**

(i) Let $N = 2$, $\mu > 0$, $\beta_-(g) < 0 < \beta_+(g)$, where

$$\beta_+(g) = \inf \left\{ b > 0 : \int_1^\infty g(t) e^{-bt} dt < \infty \right\},$$

$$\beta_-(g) = \sup \left\{ b < 0 : \int_{-\infty}^{-1} g(t) e^{bt} dt > -\infty \right\},$$

then for $\nu \in \mathcal{M}(\Omega^*; \Gamma_\mu)$ problem (3.25) admits a unique weak solution.

(ii) Let $N \geq 3$, $\mu \geq \mu_0$ and $g$ satisfy (3.12), then for $\nu \in \mathcal{M}(\Omega^*; \Gamma_\mu)$ problem (3.25) admits a unique weak solution.

• Examples: Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$ and $\nu = \sum_{n=1}^\infty a_n \delta_{e_1 \tau \n\over n}$, where $a_n > 0$ is such that $\sum_{n=1}^\infty a_n^{\tau_+} < +\infty$.

The critical exponent $\frac{N}{N-2}$ is sharp in this case.
• The case that \( \nu \geq 0 \). For \( \sigma > 0 \) small, we set \( \Omega^\sigma = \Omega \setminus \{ \overline{B}_\sigma \} \) and \( \nu^\sigma = \nu \chi_{\Omega^\sigma} \) and for \( 0 < \epsilon < \sigma \) we consider the following problem in \( \Omega^\epsilon \)

\[
\begin{cases}
L_\mu u + g(u) = \nu^\sigma & \text{in } \Omega^\epsilon, \\
u = 0 & \text{on } \partial \Omega, \\
u = 0 & \text{on } \partial B_\epsilon.
\end{cases}
\]

(3.27)

By monotonicity of \( \epsilon \mapsto u_\epsilon \) and uniform upper bound, we can pass to the limit to obtain a weak solution \( u_{\nu^\sigma} \) of

\[
L_\mu u + g(u) = \nu^\sigma \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]

(3.28)

Using monotone convergence theorem we infer that \( u_{\nu^\sigma} \to u \) in \( L^1(\Omega, |x|^{-1} d\gamma_\mu) \) and \( g(u_{\nu^\sigma}) \to g(u_\nu) \) in \( L^1(\Omega, d\gamma_\mu) \). Hence \( u = u_\nu \) is the weak solution of (3.25).

• The case that a signed measure \( \nu = \nu_+ - \nu_- \). We approximate the solution by uniform bounds and the argument of uniform integrability.
Part 4: Reduced measure

If \( k \in \mathbb{N} \), we set

\[
g_k(r) = \begin{cases} 
\min\{g(r), g(k)\} & \text{if } r \geq 0, \\
\max\{g(r), g(-k)\} & \text{if } r > 0.
\end{cases}
\]  

(3.29)

for any \( \nu \in \mathcal{M}_+(\Omega; \Gamma_\mu) \) there exists a unique weak solution \( u = u_{\nu,k} \) of

\[
\begin{cases} 
\mathcal{L}_\mu u + g_k(u) = \nu & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]  

(3.30)

Proposition

Let \( \nu \in \mathcal{M}_+(\Omega; \Gamma_\mu) \). Then the sequence of weak solutions \( \{u_{\nu,k}\} \) of

\[
\begin{cases} 
\mathcal{L}_\mu u + g_k(u) = \nu & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]  

(3.31)

decreases and converges, when \( k \to \infty \), to some nonnegative function \( u \) and there exists a measure \( \nu^* \in \mathcal{M}_+(\Omega; \Gamma_\mu) \) such that \( 0 \leq \nu^* \leq \nu \) and \( u = u_{\nu^*} \).

The ideas of the proofs

- Let \( \nu, \nu' \in \overline{\mathcal{M}}_+ (\Omega; \Gamma_\mu) \). If \( \nu' \leq \nu \) and \( \nu = \nu^* \), then \( \nu' = \nu'^* \).

- Assume that \( \nu = \nu^|_{\Omega*} + k\delta_0 \in \overline{\mathcal{M}}_+ (\Omega; \Gamma_\mu) \), then \( \nu^* = \nu^*|_{\Omega^*} + k^*\delta_0 \in \overline{\mathcal{M}}_+ (\Omega; \Gamma_\mu) \) with \( \nu^*|_{\Omega^*} \leq \nu|_{\Omega^*} \) and \( k^* \leq k \). More precisely,

  (i) If \( \mu > \mu_0 \) and \( g \) satisfies (3.11), then \( k = k^* \).
  
  (ii) If \( \mu = \mu_0 \) and \( g \) satisfies (3.13), then \( k = k^* \).
  
  (ii) If \( \mu > \mu_0 \) (resp. \( \mu = \mu_0 \)) and \( g \) does not satisfy (3.20) (resp. (3.13)), then \( k^* = 0 \).

- If \( \nu \in \overline{\mathcal{M}}_+ (\Omega; \Gamma_\mu) \), then \( \nu^* \) is the largest \( g \)-good measure smaller or equal to \( \nu \).
Outline

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4. Singular point on the boundary
• For $\mu \geq \mu_1 := -\frac{N^2}{4}$,
  \[ \int_{\mathbb{R}^N_+} |\nabla \zeta|^2 + \mu_1 \int_{\mathbb{R}^N_+} \frac{\zeta^2}{|x|^2} \, dx \geq 0 \quad \text{for all } \zeta \in C^\infty_0(\mathbb{R}^N_+). \]  
(4.1)

• $L_\mu$-harmonic functions vanishing on $\partial \mathbb{R}^N_+ \setminus \{0\}$,
  \[ \gamma_\mu(r, \sigma) = r^{\alpha_+} \psi_1(\sigma) \quad \text{and} \quad \phi_\mu(r, \sigma) = \begin{cases} r^{\alpha_-} \psi_1(\sigma) & \text{if } \mu > \mu_1, \\ r^{-\frac{N-2}{2}} \ln(r^{-1}) \psi_1(\sigma) & \text{if } \mu = \mu_1, \end{cases} \]  
(4.2)

where $\psi_1(\sigma) = \frac{x_N}{|x|}$ generates $\ker(-\Delta' + (N - 1)I)$ in $H^1_0(S^{N-1}_+)$, and where
  \[ \alpha_+ := \alpha_+(\mu) = \frac{2 - N}{2} + \sqrt{\mu + N^2/4} \quad \text{and} \quad \alpha_- := \alpha_-(\mu) = \frac{2 - N}{2} - \sqrt{\mu + N^2/4}. \]  
(4.3)

• Put $d\gamma_\mu(x) = \gamma_\mu(x) \, dx$. We define the $\gamma_\mu$-dual operator $L^*_\mu$ of $L_\mu$ by
  \[ L^*_\mu \zeta = -\Delta \zeta - \frac{2}{\gamma_\mu} \langle \nabla \gamma_\mu, \nabla \zeta \rangle \quad \text{for all } \zeta \in C^2(\overline{\mathbb{R}^N_+}), \]  
(4.4)

and we prove that $\phi_\mu$ is, in some sense, the fundamental solution of
  \[ L_\mu u = 0 \quad \text{in } \mathbb{R}^N_+, \quad u = \delta_0 \quad \text{on } \partial \mathbb{R}^N_+ \]
in the sense that
  \[ \int_{\mathbb{R}^N_+} \phi_\mu L^*_\mu \zeta \, d\gamma_\mu(x) = b_\mu \zeta(0) \quad \text{for all } \zeta \in C_c(\overline{\mathbb{R}^N_+}) \cap C^{1,1}(\mathbb{R}^N_+) \]

In a bounded domain \( \Omega \), satisfying the condition

\[
(C-1) \quad 0 \in \partial \Omega, \; \Omega \subset \mathbb{R}^N_+ \text{ and } \langle x, n \rangle = O(|x|^2) \text{ for all } x \in \partial \Omega,
\]

Hardy inequality

\[
\int_{\Omega} |\nabla \zeta|^2 + \mu_1 \int_{\Omega} \frac{\zeta^2}{|x|^2} \, dx \geq \frac{1}{4} \int_{\Omega} \frac{\zeta^2}{|x|^2 \ln^2(|x| R_\Omega^{-1})} \, dx \quad \text{for all } \zeta \in C^\infty_c(\Omega), \quad (4.5)
\]

• Let

\[
\ell^\Omega_\mu := \inf \left\{ \int_{\Omega} \left( |\nabla v|^2 + \frac{\mu}{|x|^2} v^2 \right) \, dx : v \in C^1_c(\Omega), \int_{\Omega} v^2 \, dx = 1 \right\} > 0.
\]

This first eigenvalue is achieved in \( H^1_0(\Omega) \) if \( \mu > \mu_1 \), or in the space \( H(\Omega) \) which is the closure of \( C^1_c(\Omega) \) for the norm

\[
v \mapsto \|v\|_{H(\Omega)} := \sqrt{\int_{\Omega} \left( |\nabla v|^2 + \frac{\mu_1}{|x|^2} v^2 \right) \, dx},
\]

when \( \mu = \mu_1 \). We set

\[
H_\mu(\Omega) = \begin{cases} 
H^1_0(\Omega) & \text{if } \mu > \mu_1, \\
H(\Omega) & \text{if } \mu = \mu_1.
\end{cases}
\]
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- Under the assumption \((C-1)\) the imbedding of \(H_\mu(\Omega)\) in \(L^2(\Omega)\) is compact. We denote by \(\gamma_\mu^\Omega\) the positive eigenfunction, its satisfies

\[
\mathcal{L}_\mu \gamma_\mu^\Omega = \ell_\mu \gamma_\mu^\Omega \text{ in } \Omega, \quad \gamma_\mu^\Omega = 0 \text{ on } \partial \Omega \setminus \{0\}. \tag{4.6}
\]

- there exist \(c_j = c_j(\Omega, \mu) > 0, j = 1, 2\), such that

\[
(i) \quad \gamma_\mu^\Omega(x) = c_1 \rho(x) |x|^{\alpha-1} (1 + o(1)) \quad \text{as } x \to 0,
\]

\[
(ii) \quad |\nabla \gamma_\mu^\Omega(x)| \leq c_2 \gamma_\mu^\Omega(x) / \rho(x) \quad \text{for all } x \in \Omega. \tag{4.7}
\]

- We first characterize the positive \(\mathcal{L}_\mu\)-harmonic functions which are singular at 0.

**Theorem**

Let \(\Omega\) be a \(C^2\) bounded domain such that \(0 \in \partial \Omega\) and \(\mu \geq \mu_1\). If \(u\) is a nonnegative \(\mathcal{L}_\mu\)-harmonic function in \(\Omega\) vanishing on \(B_{r_0}(0) \cap (\partial \Omega \setminus \{0\})\) for some \(r_0 > 0\), then there exists \(k \geq 0\) such that

\[
\lim_{x \to 0} \frac{u(x)}{\rho(x) |x|^{\alpha-1}} = k \quad \text{if } \mu > \mu_1
\]

and

\[
\lim_{x \to 0} \frac{|x|^{\frac{N}{2}} u(x)}{\rho(x) \ln |x|} = -k \quad \text{if } \mu = \mu_1.
\]
• Existence:

**Theorem**

Let $\Omega$ be a $C^2$ bounded domain satisfying $(C\cdot 1)$ and $\mu \geq \mu_1$. Then there exists a positive $L_\mu$-harmonic function in $\Omega$, which vanishes on $\partial \Omega \setminus \{0\}$, which satisfies

$$
\phi^\Omega_\mu(x) = \rho(x)|x|^{\alpha - 1}(1 + o(1)) \quad \text{as} \quad x \to 0,
$$

(4.8)

if $\mu > \mu_1$, and

$$
\phi^\Omega_{\mu_1}(x) = \rho(x)|x|^{-\frac{N}{2}(|\ln |x|| + 1)(1 + o(1))} \quad \text{as} \quad x \to 0,
$$

(4.9)

if $\mu = \mu_1$.

$\phi^\Omega_\mu$ is the unique function belonging to $L^1(\Omega, \rho^{-1}d\gamma^\Omega_\mu)$, which satisfies

$$
\int_\Omega uL^*_\mu \zeta d\gamma^\Omega_\mu = kc_\mu \zeta(0) \quad \text{for all} \quad \zeta \in X_\mu(\Omega),
$$

(4.10)

where $d\gamma^\Omega_\mu = \gamma^\Omega_\mu dx$, here and in the sequel the test function space

$$
X_\mu(\Omega) = \left\{ \zeta \in C(\overline{\Omega}) : \gamma^\Omega_\mu \zeta \in H_\mu(\Omega) \text{ and } \rho L^*_\mu \zeta \in L^\infty(\Omega) \right\}.
$$

Furthermore, if $u$ is a nonnegative $L_\mu$-harmonic function vanishing on $\partial \Omega \setminus \{0\}$, there exists $k \geq 0$ such that $u = k\phi^\Omega_\mu$. 

Backgrounds
Isolated singular solutions
semilinear Hardy problem
Singular point on the boundary
• Denote by $\mathcal{M}(\Omega; \gamma^\Omega_\mu)$ the set of Radon measures $\nu$ in $\Omega$ such that

$$\sup \left\{ \int_\Omega \zeta d|\lambda| : \zeta \in C_c(\Omega), 0 \leq \zeta \leq \gamma^\Omega_\mu \right\} := \int_\Omega \gamma^\Omega_\mu d|\nu| < +\infty.$$ 

If $\nu \in \mathcal{M}_+(\Omega; \gamma^\Omega_\mu)$ the measure $\gamma^\Omega_\mu \nu$ is a nonnegative bounded measure in $\Omega$. Put

$$\beta^\Omega_\mu(x) = - \frac{\partial \gamma^\Omega_\mu(x)}{\partial n_x} = \lim_{t \to 0^+} \frac{\gamma^\Omega_\mu(x - tn_x)}{t} = \lim_{t \to 0^+} \frac{\gamma^\Omega_\mu(x - tn_x)}{\rho^*(x - tn_x)}, \quad \forall x \in \partial \Omega \setminus \{0\}$$

and then

$$c_1 |x|^{\alpha + -1} \leq \beta^\Omega_\mu(x) \leq c_1 c_3 |x|^{\alpha + -1} \quad \text{for} \ x \in \partial \Omega \setminus \{0\}. \quad (4.12)$$

• Denote

$$\beta_\mu(x) = |x|^{\alpha + -1} \quad \text{for} \ x \in \mathbb{R}^N \setminus \{0\}. \quad (4.13)$$

Denote $\mathcal{M}(\partial \Omega \setminus \{0\}; \beta_\mu)$ the set Radon measures $\lambda$ in $\partial \Omega \setminus \{0\}$ such that

$$\sup \left\{ \int_{\partial \Omega \setminus \{0\}} \zeta d|\lambda| : \zeta \in C_c(\partial \Omega \setminus \{0\}), 0 \leq \zeta \leq \beta_\mu \right\} := \int_{\partial \Omega \setminus \{0\}} \beta_\mu d|\lambda| < +\infty.$$
• the existence and uniqueness of a solution to

\[
\begin{aligned}
\mathcal{L}_\mu u &= \nu & \text{in } \Omega, \\
 u &= \lambda + k\delta_0 & \text{on } \partial\Omega.
\end{aligned}
\] (4.14)

**Theorem**

Let \( \Omega \) be a \( C^2 \) bounded domain satisfying (C-1) and \( \mu \geq \mu_1 \). If \( \nu \in \mathcal{M}^+(\Omega; \gamma_\mu^\Omega) \), \( \lambda \in \mathcal{M}(\partial\Omega; \beta_\mu) \) and \( k \in \mathbb{R} \), the function

\[
uu = G_\mu^\Omega[\nu] + K_\mu^\Omega[\lambda] + k\phi_\mu^\Omega := H_\mu^\Omega[(\nu, \lambda, k)]\]

(4.15)
is the unique solution of (4.14) in the very weak sense that \( u \in L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega) \) and

\[
\int_\Omega u \mathcal{L}^*_\mu \zeta d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) + kc_\mu \zeta(0) \quad \text{for all } \zeta \in X_\mu(\Omega).
\] (4.16)

**Theorem**

Let \( \Omega \) be a \( C^2 \) bounded domain such that \( 0 \in \partial\Omega \) satisfying (C-1), \( \mu \geq \mu_1 \) and \( u \) be a nonnegative \( \mathcal{L}_\mu \)-harmonic functions in \( \Omega \). Then there exist \( \lambda \in \mathcal{M}(\partial\Omega; \beta_\mu) \) and \( k \geq 0 \), such that

\[
uu = K_\mu^\Omega[\lambda] + k\phi_\mu^\Omega = H_\mu^\Omega[(0, \lambda, k)].
\]

The couple \( (\lambda, k\delta_0) \) is called the boundary trace of \( u \).
Thank you!
Happy birthday!!!