Semilinear elliptic problems involving Leray-Hardy potential and measure data

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Workshop: Singular Problems associated to Quasilinear Equations
In celebration of Marie Francoise Bidaut-Véron and
Laurent Véron's 70th birthday

Dear Prof. Bidaut-Véron and Prof. Véron, it is a great pleasure for me to participate in this wonderful meeting to celebrate such an important birthday.

I would like to take this opportunity to express my gratitude to you for your guidance and lots of assistance. I was most fortunate to be your and Prof. Felmer's PhD student.

We will talk about

• elliptic equation with absorption nonlinearity and measure data, and elliptic equations with Hardy operators

• Isolated singular solutions of nonhomogeneous Hardy problem

$$\mathcal{L}_\mu u := -\Delta u + \frac{\mu}{|x|^2} u = f \quad \text{in} \;\; \Omega \setminus \{0\}, \qquad u = 0 \;\; \text{on} \;\; \partial\Omega$$

• semilinear Hardy equation involving measures

$$\mathcal{L}_\mu u + g(u) = \nu \quad \text{in} \quad \Omega \setminus \{0\}, \qquad u = 0 \;\; \text{on} \;\; \partial\Omega$$

• solutions of nonhomogeneous Hardy problem with the origin on the boundary

Outline

• Benilan-Brezis-Crandall, Ann Sc Norm Sup Pisa (1975); Brezis, Appl Math Opim (1984)

For $p > 1$, $f \in L^1_{loc}(\mathbb{R}^N)$, the problem

$$-\Delta u + |u|^{p-1}u = f \quad \text{in} \quad \mathbb{R}^N \tag{1.1}$$

has a unique solution u. Moreover, $u \geq 0$ if $f \geq 0$.

• Lieb-Simon, Adv. Math (1977)

The Thomas-Fermi equation, Thomas-Fermi theory of atoms, molecules

$$-\Delta u + (u-\lambda)_+^{\frac{3}{2}} = \sum_{i=1}^{l} m_i \delta_{a_i} \quad \text{in} \quad \mathbb{R}^3, \tag{1.2}$$

where $\lambda \geq 0$, $m_i > 0$ and δ_{a_i} is the Dirac mass at $a_i \in \mathbb{R}^3$. The distributional solution of (1.2) is a classical solution of

$$-\Delta u + (u-\lambda)_+^{\frac{3}{2}} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \{a_1, \cdots, a_l\}. \tag{1.3}$$

A nature question is what difference between Dirac mass source and L^1 source.

- Benilan-Brezis, *J. Evol. Eq. (2004)* (finished 1975) answered this question, when $N \geq 3$, $p \geq \frac{N}{N-2}$, $k > 0$, the problem

$$-\Delta u + |u|^{p-1}u = k\delta_0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{1.4}$$

has no solution.

- Brezis-Véron, *ARMA (1980):* when $N \geq 3$, $p \geq N/(N-2)$, the basic model

$$-\Delta u + |u|^{p-1}u = 0 \ \ \text{in} \ \ \Omega \setminus \{0\}, \qquad \text{u} = 0 \ \ \text{on} \ \ \partial\Omega \tag{1.5}$$

admits only the zero nonnegative solution.

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admits only the zero nonnegative solution.

- Veron, *NA (1981)*

For singularities of positive solutions of (1.5) for $1 < p < N/(N-2)$ $(1 < p < \infty$ if $N = 2)$, (when $(N+1)/(N-1) \le p < N/(N-2)$ the assumption of positivity is unnecessary) and that two types of singular behaviour occur:

∘ either $u(x) \sim c_N k|x|^{2-N}$ if $N \ge 3$ $u(x) \sim (-c_N k \ln|x|)$ if $N = 2$ as $|x| \to 0$ and k can take any positive value; u is said to have a *weak singularity* at 0, and actually $u = u_k$, u_k is a distributional solution of (1.4);

∘ or $u(x) \sim c_{N,p}|x|^{-\frac{2}{p-1}}$ as $x \to 0$; u is said to have a *strong singularity* at 0, and $u = u_\infty := \lim_{k\to\infty} u_k$.

- Chen-Matano-Veron, *JFA (1989): Anisotropic singularities*

When $1 < p < (N+1)/(N-1)$, u is a solution of (1.5), then

∘ either $r^{\frac{2}{p-1}} u(r,\theta) \sim \omega(\theta)$, where ω is a solution of

$$-\Delta_{\mathbb{S}^{N-1}}\omega + |\omega|^{p-1}\omega = l_p\omega \quad \text{in} \quad \mathbb{S}^{N-1};$$

∘ or there exists an integer $k < \frac{2}{p-1}$ and $\theta_0 \in [0, 2\pi)$ such that $u(r,\theta) \sim c_{N,q} k r^k \sin(k\theta + \theta_0)$ as $r = |x| \to 0$;

∘ or $u(x) \sim -c_N k \ln|x|$ as $|x| \to 0$.

- Veron, *Handb. Differ. Eq., North-Holland 2004:*

For $N \geq 3$, the problem

$$-\Delta u + g(u) = \nu \ \text{ in } \Omega, \qquad \mathrm{u} = 0 \ \text{ on } \partial\Omega \tag{1.6}$$

has a unique distributional solution u_ν if ν is a bounded Radon measure, g is nondecreasing locally Lipchitz continuous, $g(0) = 0$ and

$$\int_1^\infty (g(s) - g(-s))s^{-1-\frac{N}{N-2}}\,ds < +\infty.$$

- Vàzquez, *Proc. Royal Soc. Edinburgh. A (1983)*

When $N = 2$, introduced the exponential orders of growth of g defined by

$$\beta_{\pm}(g) = \pm \inf\left\{b > 0 : \int_1^{\infty} |g(\pm t)|e^{-bt}dt < \infty\right\} \quad (1.7)$$

if ν is any bounded measure in Ω with Lebesgue decomposition

$$\nu = \nu_r + \sum_{j\in\mathbb{N}} \alpha_j \delta_{a_j},$$

where ν_r is part of ν with no atom, $a_j \in \Omega$ and $\alpha_j \in \mathbb{R}$ satisfy

$$\frac{4\pi}{\beta_-(g)} \le \alpha_j \le \frac{4\pi}{\beta_+(g)}, \quad (1.8)$$

then

$$-\Delta u + g(u) = \nu \quad \text{in} \quad \Omega, \qquad \text{u} = 0 \quad \text{on} \quad \partial\Omega \quad (1.9)$$

admits a unique weak solution.

- Baras and Pierre , *Ann Inst Fourier Grenoble (1984)*

When $g(u) = |u|^{p-1}u$ for $p > 1$ and they discovered that if $p \ge \frac{N}{N-2}$ the problem is well posed if and only if ν is absolutely continuous with respect to the Bessel capacity $c_{2,p'}$ with $p' = \frac{p}{p-1}$.

Hardy inequalities

The Hardy inequalities

$$\frac{(N-2)^2}{4}\int_\Omega \frac{\xi^2}{|x|^2}dx \le \int_\Omega |\nabla\xi|^2 dx, \quad \forall\, \xi \in H_0^1(\Omega);$$

Improved Hardy inequality

$$\frac{(N-2)^2}{4}\int_\Omega \frac{\xi^2}{|x|^2}dx + c\int_\Omega \xi^2 dx \le \int_\Omega |\nabla\xi|^2 dx, \quad \forall\xi \in H_0^1(\Omega);$$

Denote

$$\mu_0 = -\frac{(N-2)^2}{4}.$$

Note that $\mu_0 < 0$ if $N \ge 3$ and $\mu_0 = 0$ if $N = 2$. Let Hardy operator be defined by

$$\mathcal{L}_\mu = -\Delta + \frac{\mu}{|x|^2}. \tag{1.10}$$

Singular radial solutions of $\mathcal{L}_\mu$

When $\mu \geq \mu_0$

$$\mathcal{L}_\mu u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\} \tag{1.11}$$

has two branches of radial solutions with the explicit formulas that

$$\Phi_\mu(x) = \begin{cases} |x|^{\tau_-(\mu)} & \text{if} \quad \mu < \mu_0 \\ -|x|^{\tau_-(\mu)} \ln|x| & \text{if} \quad \mu = \mu_0 \end{cases} \qquad \text{and} \qquad \Gamma_\mu(x) = |x|^{\tau_+(\mu)}, \tag{1.12}$$

where

$$\tau_-(\mu) = -\frac{N-2}{2} - \sqrt{\mu - \mu_0} \quad \text{and} \quad \tau_+(\mu) = -\frac{N-2}{2} + \sqrt{\mu - \mu_0}.$$

Here the $\tau_-(\mu)$ and $\tau_+(\mu)$ are the zero points of $\tau(\tau + N - 2) - \mu = 0$. In the following, we use the notations $\tau_- = \tau_-(\mu)$ and $\tau_+ = \tau_+(\mu)$.

semilinear Hardy problem

- Dupaigne, *JAM (2002)*

the strong, H_0^1 and distributional solutions of

$$\mathcal{L}_\mu u = u^p + tf, \;\; u > 0 \;\text{ in } \Omega, \qquad u = 0 \;\text{ on }\; \partial\Omega. \tag{1.13}$$

∘ a classical solution u is a $C^2(\bar{\Omega} \setminus \{0\})$ function verifies the equation pointwise in $\Omega \setminus \{0\}$ and $u(x) \le c\Gamma_\mu$ for some $c > 0$;

∘ a H^1 solution u is a $H_0^1(\Omega)$ function verifies the identity

$$\int_\Omega (\nabla u \nabla \xi - \frac{\mu}{|x|^2} u\xi) = \int_\Omega (u^p + tf)\xi, \quad \forall \xi \in H_0^1(\Omega);$$

∘ a distributional solution u, if $u \in L^1(\Omega)$, $\frac{u}{|x|^2} \in L^1(\Omega, \rho dx)$ and u verifies that

$$\int_\Omega u \mathcal{L}_\mu \xi = \int_\Omega (u^p + tf)\xi, \quad \forall \xi \in C^2(\bar{\Omega}) \cap C_0(\Omega),$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$.

Dupaigne's main results

Theorem

Assume that $N \geq 3$, $\mu \in [\mu_0, 0)$, f *is a smooth, bounded and nonnegative function and*

$$q^*_\mu = 1 + \frac{2}{-\tau_+(\mu)}$$

For $1 < p < q^*_\mu$, *there exists* t_0 *such that*

(i) *if* $0 < t < t_0$, *problem (1.13) has a minimal classical solution;*

(ii) *if* $t = t_0$, *problem (1.13) has a minimal distributional solution;*

(iii) *if* $t > t_0$, *problem (1.13) has no distributional solution.*

- Brezis-Dupaigne-Tesei *Sel Math (2005)*

When $t = 0$, (1.13) has a nontrivial nonnegative solution of for $p < q^*_\mu$ and does not have nonnegative distributional solutions for $p \geq q^*_\mu$.

- Guerch and Véron, *Rev mat Iberoamericana 1991*

○ $\mu > \mu_0$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(0) \geq 0$

$$\int_1^\infty (g(s) - g(-s))s^{-1-\frac{\tau_- - 2}{\tau_-}} ds < \infty; \tag{1.14}$$

○ $\mu = \mu_0$, $k > 0$, $N \geq 3$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(0) \geq 0$ and

$$\int_1^\infty g\left(kt^{\frac{N-2}{N+2}} \ln t\right) t^{-2} dt < \infty, \tag{1.15}$$

semilinear Hardy problem

$$\mathcal{L}_\mu u + g(u) = 0 \ \text{ in } \ \Omega \setminus \{0\}, \quad u = 0 \ \text{ on } \ \partial\Omega \tag{1.16}$$

has a classical solution $u_k \in C^2(\bar{\Omega} \setminus \{0\})$ such that $\lim_{|x| \to 0} \frac{u_k(x)}{\Phi_\mu(x)} = k$.

- Cîrstea, *American mathematical society 2014*

The positive solution of semilinear Hardy equation $\mathcal{L}_\mu u + g(u) = 0$ in $\Omega \setminus \{0\}$ has three possible singularities at the origin:

$$\text{either} \quad \lim_{x\to 0} \frac{u(x)}{\Phi_\mu(x)} = +\infty \quad \text{or} \quad \lim_{x\to 0} \frac{u(x)}{\Phi_\mu(x)} \in (0, +\infty), \tag{1.17}$$

$$\text{or} \qquad \lim_{x\to 0} \frac{u(x)}{\Gamma_\mu(x)} \in (0, +\infty). \tag{1.18}$$

Related elliptic problem with boundary Hardy potential:

- Gkikas-Véron, *NA 2015*
- Nguyen, *CVPDE 2017:*
- Marcus-Nguyen, *Math Ann 2019;*
- Bandle-Marcus-Moroz, *Israel Journal of Mathematics 2017*

Some questions

• When $\mu = 0$, $\Phi_0(x) = |x|^{2-N}$ if $N \geq 3$ and $\Gamma_\mu = 1$, function Φ_0 verifies the distributional identity

$$\int_{\mathbb{R}^N} \Phi_0 \mathcal{L}_0 \xi \, dx = c_0 \xi(0), \quad \forall \xi \in C_c^2(\mathbb{R}^N)$$

• For $\mu \in [\mu_0, 0)$, there holds that

$$\int_{\mathbb{R}^N} \Phi_\mu \mathcal{L}_\mu \xi \, dx = \int_{\mathbb{R}^N} \Gamma_\mu \mathcal{L}_\mu \xi \, dx = 0, \quad \forall \xi \in C_c^2(\mathbb{R}^N) \tag{1.19}$$

For $\mu \in [\mu_0, 0)$, the Dirac mass can not be used to express the singularities of the function Φ_μ or Γ_μ in the traditional distributional sense.

• Especially, when $\mu > 0$ large enough, the distributional identity (1.19) for Φ_μ is not well-defined.

Outline

New distributional identity

When $\mu \geq \mu_0$, Φ_μ and Γ_μ satisfy $\mathcal{L}_\mu u = 0$ in $\mathbb{R}^N \setminus \{0\}$.

Theorem

Let $d\gamma_\mu(x) = \Gamma_\mu(x)dx$ *and*

$$\mathcal{L}^*_\mu = -\Delta - 2\frac{\tau_+(\mu)}{|x|^2}\, x \cdot \nabla. \tag{2.1}$$

Then

$$\int_{\mathbb{R}^N} \Phi_\mu \mathcal{L}^*_\mu(\xi)\, d\gamma_\mu = c_\mu \xi(0), \quad \forall\, \xi \in C^2_c(\mathbb{R}^N), \tag{2.2}$$

where

$$c_\mu = \begin{cases} 2\sqrt{\mu - \mu_0}\,|\mathbb{S}^{N-1}| & \text{if} \quad \mu > \mu_0, \\ |\mathbb{S}^{N-1}| & \text{if} \quad \mu = \mu_0. \end{cases} \tag{2.3}$$

• H. Chen, A. Quaas and F. Zhou, On nonhomogeneous elliptic equations with the Hardy-Leray potentials, *Accepted by JAM, arXiv:1705.08047*.

• In fact we show that

$$\Gamma_\mu \cdot \mathcal{L}_\mu(\Phi_\mu) = c_\mu \delta_0. \tag{2.4}$$

In particular, for $\mu = 0$, $\Gamma_\mu = 1$, $\mathcal{L}^*_\mu = -\Delta$ and (2.4) reduces to

$$-\Delta \Phi_0 = c_0 \delta_0.$$

• Observation: $\tau_-(\mu) + \tau_+(\mu) = 2 - N$, for $\xi \in C^2_c(\mathbb{R}^N)$, we use test function $\Gamma_\mu \xi$,

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N \setminus \overline{B_r(0)}} \mathcal{L}_\mu(\Phi_\mu)\Gamma_\mu \xi \, dx \\
&= \int_{\mathbb{R}^N \setminus \overline{B_r(0)}} \Phi_\mu \mathcal{L}^*_\mu(\xi) \, d\gamma_\mu + \int_{\partial B_r(0)} \left(\nabla \Phi_\mu \cdot \frac{x}{|x|} \Gamma_\mu - \nabla \Gamma_\mu \cdot \frac{x}{|x|} \Phi_\mu \right) \xi \, d\omega \\
&\quad - \int_{\partial B_r(0)} \Phi_\mu \Gamma_\mu \left(\nabla \xi \cdot \frac{x}{|x|} \right) d\omega.
\end{aligned}$$

• Here Φ_μ is said to be a fundamental solution of $\mathcal{L}_\mu$. We note that the fundamental solution Φ_μ keeps positive when $\mu < \mu_0$ and changes signs for $\mu = \mu_0$.

Bounded domain

In the bounded C^2 domain Ω containing the origin,

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \lim_{x \to 0} u(x)\Phi_\mu^{-1}(x) = 1 \end{cases} \tag{2.5}$$

has a unique solution $\Phi_{\mu,\Omega}$.

Theorem

Let $\Phi_{\mu,\Omega}$ be the solution of (2.5), then

$$\int_\Omega \Phi_{\mu,\Omega} \mathcal{L}_\mu^*(\xi)\, d\gamma_\mu = c_\mu \xi(0), \quad \forall\, \xi \in C_0^{1.1}(\Omega). \tag{2.6}$$

Approximation of the fundamental solution

Let $\{\delta_n\}_n$ be a sequence of nonnegative L^∞ functions that supp $\delta_n \subset B_{r_n}(0)$, where $r_n \to 0$ as $n \to +\infty$,

$$\delta_n \to \delta_0 \quad \text{as} \quad n \to +\infty \;\text{ in the distributional sense.}$$

For any n, the problem

$$\begin{cases} \mathcal{L}_\mu u = c_\mu \delta_n / \Gamma_\mu & \text{in} \quad \Omega \setminus \{0\}, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \\ \lim\limits_{x \to 0} u(x) \Phi_\mu^{-1}(x) = 0 \end{cases} \tag{2.7}$$

has a unique solution w_n.
Then

$$\lim_{n \to +\infty} w_n(x) = \Phi_{\mu,\Omega}(x), \quad \forall\, x \in \Omega \setminus \{0\}.$$

We consider nonhomogeneous problem

$$\mathcal{L}_\mu u = f \ \text{ in } \Omega \setminus \{0\}, \qquad u = 0 \ \text{ on } \partial\Omega. \tag{2.8}$$

Theorem

Let $\mu \geq \mu_0$, f *be a function in* $C^\theta_{loc}(\overline{\Omega} \setminus \{0\})$ *for some* $\theta \in (0,1)$.
(i) *Assume that*

$$\int_\Omega |f| \, d\gamma_\mu < +\infty, \tag{2.9}$$

then problem (2.8), subject to $\lim\limits_{x \to 0} u(x)\Phi_\mu^{-1}(x) = k$ *with* $k \in \mathbb{R}$, *has a unique solution* u_k, *which satisfies the distributional identity*

$$\int_\Omega u_k \mathcal{L}^*_\mu(\xi) \, d\gamma_\mu = \int_\Omega f\xi \, d\gamma_\mu + c_\mu k \xi(0), \quad \forall \, \xi \in C_0^{1.1}(\Omega). \tag{2.10}$$

(ii) *Assume that* f *verifies (2.9) and* u *is a nonnegativesolution of (2.8), then* u *satisfies (2.10) for some* $k \geq 0$.

(iii) *Assume that* $f \geq 0$ *and*

$$\lim_{r \to 0^+} \int_{\Omega \setminus B_r(0)} f \, d\gamma_\mu = +\infty, \tag{2.11}$$

then problem (2.8) has no nonnegativesolutions.

Part 1: existence for $f \in L^1(\Omega, d\gamma_\mu)$

Lemma

Assume that $f \in C^\theta(\bar{\Omega})$ for some $\theta \in (0,1)$, then

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in } \ \Omega \setminus \{0\}, \\ u = 0 & \text{on } \ \partial\Omega, \\ \lim\limits_{x \to 0} u(x)\Phi_\mu^{-1}(x) = 0 \end{cases} \tag{2.12}$$

has a unique solution u_f satisfying the distributional identity:

$$\int_\Omega u_f \mathcal{L}_\mu^*(\xi)\, d\gamma_\mu = \int_\Omega f\xi\, d\gamma_\mu, \quad \forall\, \xi \in C_0^{1.1}(\Omega). \tag{2.13}$$

• *The case $\mu > \mu_0$.* Indeed, for $\mu > \mu_0$, we can choose $\tau_0 \in (\tau_-(\mu), \min\{2, \tau_+(\mu)\})$, and denote

$$V_0(x) = |x|^{\tau_0}, \quad \forall\, x \in \Omega \setminus \{0\}.$$

Then

$$\mathcal{L}_\mu V_0(x) = c_{\tau_0}|x|^{\tau_0 - 2},$$

where $c_{\tau_0} = \mu - \tau_0(\tau_0 + N - 2) > 0$.

Since f is bounded, there exists $t_0 > 0$ such that

$$|f(x)| \le t_0 c_{\tau_0} |x|^{\tau_0 - 2}, \qquad \forall x \in \Omega \setminus \{0\},$$

then $t_0 V_0$ and $-t_0 V_0$ are supersolution and subsolution of (2.12) respectively.

• *The case $\mu = \mu_0$ and $N \ge 3$.*
∘ $\mu \mapsto u_\mu$ is decreasing in $[\mu_0, 0)$.
∘ a uniformly bound for u_μ for $\mu > \mu_0$

$$V(x) = |x|^{\tau_+(\mu_0)} - (s_0|x|)^2, \quad \forall x \in \Omega \setminus \{0\},$$

where $s_0 > 0$ and $V > 0$ in $\Omega \setminus \{0\}$. Then there exists $t_0 > 0$ such that

$$u_\mu \le t_0 V \quad \text{in} \quad \Omega \setminus \{0\}.$$

For $\xi \in C_0^{1.1}(\Omega)$, there exists $c > 0$ independent of μ such that

$$|\mathcal{L}_\mu^*(\xi)| \le c\|\xi\|_{C_0^{1.1}(\Omega)} + |\mu|\|\xi\|_{C_0^1(\Omega)}|x|^{-1}.$$

∘ From the dominate monotonicity convergence theorem, there exists $u_{\mu_0} \le t_0 V$ such that

$$u_\mu \to u_{\mu_0} \quad \text{as} \quad \mu \to \mu_0^+ \quad \text{a.e. in} \ \ \Omega \quad \text{and in} \ \ L^1(\Omega, |x|^{-1} d\gamma_\mu)$$

and

$$\int_\Omega u_{\mu_0} \mathcal{L}_{\mu_0}^*(\xi)\, d\gamma_{\mu_0} = \int_\Omega f\, \xi d\gamma_{\mu_0}$$

Part 2: nonexistence for $f \notin L^1(\Omega, d\gamma_\mu)$

• From (2.11) and the fact $f \in C^\theta(\overline{\Omega} \setminus \{0\})$, for any r_n, we have that

$$\lim_{r\to 0^+} \int_{B_{r_n}(0)\setminus B_r(0)} f(x) d\gamma_\mu = +\infty,$$

then there exists $R_n \in (0, r_n)$ such that $\int_{B_{r_n}(0)\setminus B_{R_n}(0)} f d\gamma_\mu = n$.
Let $\delta_n = \frac{1}{n}\Gamma_\mu f \chi_{B_{r_n}(0)\setminus B_{R_n}(0)}$, then the problem

$$\begin{cases} \mathcal{L}_\mu u \cdot \Gamma_\mu = \delta_n & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \lim\limits_{x\to 0} u(x)\Phi_\mu^{-1}(x) = 0 \end{cases}$$

has a unique positive solution w_n satisfying

$$\int_\Omega w_n \mathcal{L}_\mu(\Gamma_\mu \xi) dx = \int_\Omega \delta_n \xi dx, \quad \forall \xi \in C_0^{1.1}(\Omega).$$

• For any $\xi \in C_0^{1.1}(\Omega)$, we have that

$$\int_\Omega w_n \mathcal{L}_\mu^*(\xi)\, d\gamma_\mu = \int_\Omega \delta_n \xi\, dx \to \xi(0) \quad \text{as} \quad n \to +\infty.$$

Therefore for any compact set $K \subset \Omega \setminus \{0\}$,

$$\|w_n - \Phi_{\mu,\Omega}\|_{C^1(K)} \to 0 \quad \text{as} \quad n \to +\infty.$$

Fix $x_0 \in \Omega \setminus \{0\}$ and $r_0 = \frac{\min\{|x_0|,\, \rho(x_0)\}}{2}$ and $K = \overline{B_{r_0}(x_0)}$, then there exists $n_0 > 0$ such that for $n \geq n_0$,

$$w_n \geq \frac{1}{2}\Phi_{\mu,\Omega} \quad \text{in} \quad K. \tag{2.14}$$

• Let u_n be the solution of

$$\begin{cases} \mathcal{L}_\mu u \cdot \Gamma_\mu = n\delta_n \qquad \text{in} \quad \Omega \setminus \{0\}, \\ \qquad u = 0 \qquad \text{on} \quad \partial\Omega, \\ \lim\limits_{x \to 0} u(x)\Phi_\mu^{-1}(x) = 0, \end{cases}$$

thus, together with (2.14), we have that

$$u_n \geq n w_n \geq \frac{n}{2}\Phi_{\mu,\Omega} \ \text{ in } \ K$$

and

$$u_f(x_0) \geq u_n(x_0) \to +\infty \quad \text{as} \quad n \to +\infty,$$

which contradicts that u_f is classical solution of (2.8)

Part 3: nonexistence when $\mu < \mu_0$

Theorem

Assume that $\mu < \mu_0$ and f is a measurable nonnegative function, then problem (2.8) has no nontrivial nonnegative solutions.

Sketch of the proof. Let u_0 be a nontrivial nonnegative solution of (2.8).

$$\mathcal{L}_{\mu_0} u_0 = (\mu_0 - \mu)\frac{u_0}{|x|^2} + f \geq (\mu_0 - \mu)\epsilon_0 \frac{\chi_{B_{r_0}(x_0)}}{|x|^2},$$

When $N \geq 3$, for $x \in B_{r_0}(0) \setminus \{0\}$,

$$u_0(x) \geq (\mu_0 - \mu)\epsilon_0 \mathbb{G}_{\mu_0}[\chi_{B_{r_0}(x_0)}] \geq c_0 |x|^{-\frac{N-2}{2}},$$

then

$$\begin{aligned}\int_{\Omega\setminus B_r(0)} [(\mu_0 - \mu)\frac{u_0}{|x|^2} + f] d\gamma_{\mu_0} &\geq c_0 \int_{B_{r_0}(0)\setminus B_r(0)} |x|^{-N} dx \\ &\to +\infty \quad \text{as} \quad r \to 0^+.\end{aligned}$$

We obtain that

$$\mathcal{L}_\mu u = (\mu_0 - \mu)\frac{u_0}{|x|^2} + f \ \text{ in } \Omega \setminus \{0\}, \quad u = 0 \ \text{ on } \partial\Omega \tag{2.15}$$

has no nonnegative solution.

Outline

The nonlinear Poisson equation

$$\mathcal{L}_\mu u + g(u) = \nu \ \text{ in } \ \Omega, \qquad \text{u} = 0 \ \text{ on } \ \partial\Omega, \tag{3.1}$$

where $\mu \geq \mu_0$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(0) \geq 0$ and ν is a Radon measure in Ω.

• we denote by $\mathfrak{M}(\Omega^*; \Gamma_\mu)$, the set of Radon measures ν in Ω^* such that

$$\int_{\Omega^*} \Gamma_\mu d|\nu| := \sup\left\{ \int_{\Omega^*} \zeta d|\nu| \ : \zeta \in C_c(\Omega^*),\ 0 \leq \zeta \leq \Gamma_\mu \right\} < \infty, \tag{3.2}$$

where $\Omega^* = \Omega \setminus \{0\}$.

• *we denote by* $\mathfrak{M}(\Omega; \Gamma_\mu)$ *the set of measures* ν *on* Ω *which coincide with the above natural extension of* $\nu\lfloor_{\Omega^*} \in \mathfrak{M}_+(\Omega^*; \Gamma_\mu)$. If $\nu \in \mathfrak{M}_+(\Omega; \Gamma_\mu)$ we define the measure $\Gamma_\mu \nu$ in the following way

$$\int_\Omega \zeta d(\Gamma_\mu \nu) = \sup\left\{ \int_{\Omega^*} \eta \Gamma_\mu d\nu \ : \eta \in C_c(\Omega^*)\,, 0 \leq \eta \leq \zeta \right\} \text{ for all } \zeta \in C_c(\Omega)\,,\ \zeta \geq 0. \tag{3.3}$$

• We denote by $\overline{\mathfrak{M}}(\Omega;\Gamma_\mu)$ the set of measures which can be written under the form

$$\nu = \nu\lfloor_{\Omega^*} + k\delta_0, \tag{3.4}$$

where $\nu\lfloor_{\Omega^*} \in \mathfrak{M}(\Omega;\Gamma_\mu)$ and $k \in \mathbb{R}$.

• We denote $\overline{\Omega}^* := \overline{\Omega} \setminus \{0\}$, $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ and

$$\mathbb{X}_\mu(\Omega) = \Big\{\xi \in C_0(\overline{\Omega}) \cap C^1(\overline{\Omega}^*) : |x|\mathcal{L}_\mu^*\xi \in L^\infty(\Omega)\Big\}. \tag{3.5}$$

Clearly, $C_0^{1,1}(\overline{\Omega}) \subset \mathbb{X}_\mu(\Omega)$.

Definition

• We say that u is a weak solution of (3.1) with $\nu \in \overline{\mathfrak{M}}(\Omega;\Gamma_\mu)$ such that $\nu = \nu\lfloor_{\Omega^*} + k\delta_0$ if $u \in L^1(\Omega, |x|^{-1}d\gamma_\mu)$, $g(u) \in L^1(\Omega, \rho d\gamma_\mu)$ and

$$\int_\Omega \left[u\mathcal{L}_\mu^*\xi + g(u)\xi\right]\, d\gamma_\mu = \int_\Omega \xi d(\Gamma_\mu\nu) + c_\mu k\xi(0) \quad \text{for all } \ \xi \in \mathbb{X}_\mu(\Omega). \tag{3.6}$$

• the Dirac mass at 0 does not belong to $\mathfrak{M}(\Omega;\Gamma_\mu)$ although it is a limit of $\{\nu_n\} \subset \mathfrak{M}(\Omega;\Gamma_\mu)$.

Definition

• A continuous function $g:\mathbb{R}\to\mathbb{R}$ such that $rg(r)\geq 0$ for all $r\in\mathbb{R}$ satisfies the weak Δ_2-condition if there exists a positive nondecreasing function $t\in\mathbb{R}\mapsto K(t)$ such that

$$|g(s+t)| \leq K(t)\left(|g(s)|+|g(t)|\right) \quad \text{for all } (s,t)\in\mathbb{R}\times\mathbb{R} \text{ s.t. } st\geq 0. \tag{3.7}$$

It satisfies the Δ_2-condition if the above function K is constant.

• Critical exponent

$$p^*_\mu = 1-\frac{2}{\tau_-}. \tag{3.8}$$

Note that $p^*_\mu < p^*_0$ if $\mu>0$ and $p^*_\mu > p^*_0$ if $\mu<0$.

• H. Chen and L. Véron, Weak solutions of semilinear elliptic equations with Leray-Hardy potential and measure data, *Mathematics in Engineering 1*, (2019).

Theorem

Let $\mu > 0$ *if* $N = 2$, $\mu \geq \mu_0$ *if* $N \geq 3$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be a Hölder continuous nondecreasing function such that* $g(0) = 0$. *Then for any* $\nu \in L^1(\Omega, d\gamma_\mu)$, *problem (3.1) has a unique weak solution* u_ν *such that for some* $c_1 > 0$,

$$\|u_\nu\|_{L^1(\Omega,|x|^{-1}d\gamma_\mu)} \leq c_1 \|\nu\|_{L^1(\Omega,d\gamma_\mu)}.$$

Furthermore, if $u_{\nu'}$ *is the solution of (3.1) with right-hand side* $\nu' \in L^1(\Omega, d\gamma_\mu)$, *there holds*

$$\int_\Omega \left[|u_\nu|\mathcal{L}_\mu^*\xi + |g(u_\nu))|\xi\right] d\gamma_\mu \leq \int_\Omega (\nu)\text{sgn}(u_\nu)\xi d\gamma_\mu \tag{3.9}$$

and

$$\int_\Omega \left[(u_\nu)_+\mathcal{L}_\mu^*\xi + (g(u_\nu))_+\xi\right] d\gamma_\mu \leq \int_\Omega \nu\text{sgn}_+(u_\nu)\xi d\gamma_\mu \tag{3.10}$$

for all $\xi \in \mathbb{X}_\mu(\Omega)$, $\xi \geq 0$, *where* $\text{sgn}(t) = 1$ *if* $t > 0$, $\text{sgn}(0) = 0$ *and* $\text{sgn}(t) = -1$ *if* $t < 0$.

• Remark: (3.9) and (3.10) are Kato's type Inequalities; these inequalities plays an important role in the derivation of uniqueness.

Now we state the existence of weak solution in the subcritical case with $\mu > \mu_0$.

Theorem

Let $\mu > \mu_0$ and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $g(r)r \ge 0$ for any $r \in \mathbb{R}$. If g satisfies the weak Δ_2-condition and

$$\int_1^\infty (g(s) - g(-s))s^{-1-\min\{p^*_\mu, p^*_0\}} ds < \infty. \tag{3.11}$$

Then for $\nu \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ problem (3.1) admits a unique weak solution u_ν. Furthermore, the mapping: $\nu \mapsto u_\nu$ is increasing.

• For $\nu = \nu\lfloor_{\Omega^*} + c_\mu k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ and $g(t) = |t|^{p-1}t$, problem (3.1) has a unique solution if

(i) $1 < p < p^*_\mu$ in the case $\nu\lfloor_{\Omega^*} = 0$;

(ii) $1 < p < p^*_0$ in the case $k = 0$;

(iii) $1 < p < \min\{p^*_\mu, p^*_0\}$ in the case $k \ne 0$ and $\nu\lfloor_{\Omega^*} \ne 0$.

• Examples: Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$ and $\nu = \sum_{n=1}^\infty a_n\delta_{\frac{e_1}{n}} + k\delta_0$, where $a_n > 0$ is such that $\sum_{n=1}^\infty a_n^{\tau_+} < +\infty$.

Theorem

Assume that $N \geq 3$, $\mu = \mu_0$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(r)r \geq 0$ for any $r \in \mathbb{R}$ satisfying the weak Δ_2-condition and

$$\int_1^{+\infty} (g(s) - g(-s))s^{-1-\frac{N}{N-2}} ds < +\infty. \tag{3.12}$$

Then for any $\nu = \nu\lfloor_{\Omega^*} + c_\mu k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ problem (3.1) admits a unique weak solution u_ν.
Furthermore, if $\nu\lfloor_{\Omega^*} = 0$, condition (3.12) can be replaced by the following weaker one

$$\int_1^{\infty} (g(t) - g(-t)) (\ln t)^{\frac{N+2}{N-2}} t^{-1-\frac{N+2}{N-2}} dt < \infty. \tag{3.13}$$

• Examples: $\nu = k\delta_0$ and $g(t) = t^{\frac{N+4}{N-2}} (\ln t)^\tau$ with $\tau > \frac{2N}{N-2}$, (3.1) has an isolated singular solution $u_k > 0$.

In the supercritical case, we set $g_p(u) = |u|^{p-1}u$, i.e.

$$\mathcal{L}_\mu u + g_p(u) = \nu \;\; \text{in} \;\; \Omega, \quad \text{u} = 0 \;\; \text{on} \;\; \partial\Omega, \tag{3.14}$$

Theorem

Assume that $N \geq 3$. Then $\nu = \nu\lfloor_{\Omega^*} \in \mathfrak{M}(\Omega; \Gamma_\mu)$ is g_p-good if and only if for any $\epsilon > 0$, $\nu_\epsilon = \nu\chi_{B_\epsilon^c}$ is absolutely continuous with respect to the $c_{2,p'}$-Bessel capacity.

Finally we characterize the compacts removable sets in Ω.

Theorem

Assume that $N \geq 3$, $p > 1$ and K is a compact set of Ω. Then any weak solution of

$$\mathcal{L}_\mu u + g_p(u) = 0 \quad \text{in} \;\; \Omega \setminus K \tag{3.15}$$

can be extended a solution of the same equation in whole Ω if and only if

(i) $c_{2,p'}(K) = 0$ if $0 \notin K$;

(ii) $p \geq p_\mu^*$ if $K = \{0\}$;

(iii) $c_{2,p'}(K) = 0$ if $\mu \geq 0$, $0 \in K$ and $K \setminus \{0\} \neq \{\emptyset\}$;

(iv) $c_{2,p'}(K) = 0$ and $p \geq p_\mu^*$ if $\mu < 0$, $0 \in K$ and $K \setminus \{0\} \neq \{\emptyset\}$.

Part 1: linear problem

Lemma

If $\nu \in \overline{\mathfrak{M}}(\Omega; \Gamma_\mu)$, *then*

$$\mathcal{L}_\mu u = \nu \quad \text{in } \Omega, \qquad \mathrm{u} = 0 \quad \text{on } \partial\Omega. \tag{3.16}$$

admits a unique solution in $L^1(\Omega, |x|^{-1} d\gamma_\mu)$, *denoted by* $\mathbb{G}_\mu[\nu]$, *and this defines the Green operator of* $\mathcal{L}_\mu$ *in* Ω *with homogeneous Dirichlet conditions.*

- Let $\{\nu_n\} \subset L^1(\Omega, \rho d\gamma_\mu)$ be a sequence such that $\nu_n \geq 0$ and

$$\int_\Omega \xi \Gamma_\mu \nu_n dx \to \int_\Omega \xi d(\Gamma_\mu \nu) \quad \text{for all } \xi \in \mathbb{X}_\mu(\Omega),$$

with $n \in \mathbb{N}$, the weak solution of

$$\mathcal{L}_\mu u_n = \nu_n \text{ in } \Omega, \qquad \mathrm{u_n} = 0 \quad \text{on } \partial\Omega \tag{3.17}$$

satisfies that for any open sets O verifying $\bar{O}\Omega \setminus B_\epsilon(0)$ for some $c > 0$ independent of n but dependent of O',

$$\|u_n\|_{W^{1,q}(O)} \leq c \|\nu\|_{\mathfrak{M}(\Omega, \Gamma_\mu)} .$$

That is, $\{u_n\}$ is uniformly bounded in $W^{1,q}_{loc}(\Omega \setminus \{0\})$.

- Let $\omega \subset \Omega$ be a Borel set and the solution ψ_ω of

$$\begin{cases} \mathcal{L}_\mu^* \psi_\omega = |x|^{-1}\chi_\omega & \text{in } \Omega, \\ \psi_\omega = 0 & \text{on } \partial\Omega \end{cases} \tag{3.18}$$

has the property

$$\lim_{|\omega|\to 0} \psi_\omega(x) = 0 \quad \text{uniformly in } B_1$$

and

$$\int_\omega \frac{u_n}{|x|} d\gamma_\mu(x) = \int_\omega \nu_n \Gamma_\mu \psi_\omega dx \le \sup_\Omega \psi_\omega \int_\omega \nu_n \Gamma_\mu dx \to 0 \ \text{ as } \ |\omega| \to 0.$$

This shows that $\{u_n\}$ is uniformly integrable for the measure $|x|^{-1} d\gamma_\mu$.

Part 2: Isolated singular solutions

Lemma

Let $k \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $rg(r) \geq 0$ for all $r \in \mathbb{R}$. Then problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = k\delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.19}$$

admits a unique solution $u := u_{k\delta_0}$ if one of the following conditions is satisfied:

(i) $N = 2$, $\mu > \mu_0$ and g satisfies

$$\int_1^\infty (g(s) - g(-s))\, s^{-1-p^*_\mu} ds < \infty; \tag{3.20}$$

(ii) $N \geq 3$, $\mu = \mu_0$ and g satisfies (3.13).

• For $\mu > \mu_0$ [Guerch-Veron 1991] for any $k \in \mathbb{R}$ there exists a radial function $v_{k,1}$ (resp. $v_{k,R}$) defined in B_1^* (resp. B_R^*) satisfying

$$\mathcal{L}_\mu v + g(v) = 0 \qquad \text{in } B_1^* \quad (\text{resp. in } B_R^*), \tag{3.21}$$

vanishing respectively on ∂B_1 and ∂B_R and satisfying

$$\lim_{x\to 0} \frac{v_{k,1}(x)}{\Phi_\mu(x)} = \lim_{x\to 0} \frac{v_{k,R}(x)}{\Phi_\mu(x)} = \frac{k}{c_\mu}. \tag{3.22}$$

• For $\mu = \mu_0$, [Guerch-Veron 1991] shows the existence of isolated singular solution if for some $b > 0$ there holds

$$I := \int_1^\infty g\left(bt^{\frac{N-2}{N+2}}\ln t\right)t^{-2}dt < \infty, \tag{3.23}$$

set $s = t^{\frac{N-2}{N+2}}$ and $\beta = \frac{N+2}{N-2}b$, then

$$I = \frac{N+2}{N-2}\int_1^\infty g\left(\beta s\ln s\right)s^{-\frac{2N}{N-2}}ds$$

Set $\tau = \beta s\ln s$, then

$$\ln\tau = \ln s\left(1 + \frac{\ln\ln s}{\ln s} + \frac{\ln\beta}{\ln s}\right) \Longrightarrow \ln s = \ln\tau(1+o(1)) \quad \text{as} \;\; s\to\infty.$$

We infer that for $\epsilon > 0$ there exists $s_\epsilon > 2$ and $\tau_\epsilon = s_\epsilon \ln s_\epsilon$ such that

$$(1-\epsilon)\beta^{\frac{N+2}{N-2}} \le \frac{\displaystyle\int_{s_\epsilon}^\infty g\left(\beta s\ln s\right)s^{-\frac{2N}{N-2}}ds}{\displaystyle\int_{\tau_\epsilon}^\infty g\left(\tau\right)\left(\ln\tau\right)^{\frac{N+2}{N-2}}\tau^{-\frac{2N}{N-2}}d\tau} \le (1+\epsilon)\beta^{\frac{N+2}{N-2}}. \tag{3.24}$$

Thus, $I < +\infty$ is equivalent to (3.13).

Part 3: Measures in Ω^*

$$\mathcal{L}_\mu u + g(u) = \nu \text{ in } \Omega, \qquad \text{u} = 0 \quad \text{on } \partial\Omega \tag{3.25}$$

Lemma

(i) Let $N = 2$, $\mu > 0$, $\beta_-(g) < 0 < \beta_+(g)$, where

$$\begin{aligned} \beta_+(g) &= \inf\left\{b > 0 : \int_1^{\infty} g(t)\, e^{-bt} dt < \infty\right\}, \\ \beta_-(g) &= sup\left\{b < 0 : \int_{-\infty}^{-1} g(t)\, e^{bt} dt > -\infty\right\}, \end{aligned} \tag{3.26}$$

then for $\nu \in \mathfrak{M}(\Omega^*; \Gamma_\mu)$ problem (3.25) admits a unique weak solution.
(ii) Let $N \geq 3$, $\mu \geq \mu_0$ and g satisfy (3.12), then for $\nu \in \mathfrak{M}(\Omega^*; \Gamma_\mu)$ problem (3.25) admits a unique weak solution.

• Examples: Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$ and $\nu = \sum_{n=1}^{\infty} a_n \delta_{\frac{e_1}{n}}$, where $a_n > 0$ is such that $\sum_{n=1}^{\infty} a_n^{\tau_+} < +\infty$.
The critical exponent $\frac{N}{N-2}$ is sharp in this case.

• The case that $\nu \geq 0$. For $\sigma > 0$ small, we set $\Omega^\sigma = \Omega \setminus \{\overline{B}_\sigma\}$ and $\nu_\sigma = \nu\chi_{\Omega^\sigma}$ and for $0 < \epsilon < \sigma$ we consider the following problem in Ω^ϵ

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\sigma & \text{in } \Omega^\epsilon, \\ u = 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial \mathrm{B}_\epsilon. \end{cases} \tag{3.27}$$

By monotonicity of $\epsilon \mapsto u_\epsilon$ and uniform upper bound, we can pass to the limit to obtain a weak solution u_{ν_σ} of

$$\mathcal{L}_\mu u + g(u) = \nu_\sigma \ \text{ in } \Omega, \qquad \mathrm{u} = 0 \quad \text{on } \partial\Omega. \tag{3.28}$$

Using monotone convergence theorem we infer that $u_{\nu_\sigma} \to u$ in $L^1(\Omega, |x|^{-1} d\gamma_\mu)$ and $g(u_{\nu_\sigma}) \to g(u_\nu)$ in $L^1(\Omega, d\gamma_\mu)$. Hence $u = u_\nu$ is the weak solution of (3.25).

• The case that a signed measure $\nu = \nu_+ - \nu_-$. We approximate the solution by uniform bounds and the argument of uniform integrability.

Part 4: Reduced measure

If $k \in \mathbb{N}$, we set

$$g_k(r) = \begin{cases} \min\{g(r), g(k)\} & \text{if } r \geq 0, \\ \max\{g(r), g(-k)\} & \text{if } r > 0. \end{cases} \tag{3.29}$$

for any $\nu \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ there exists a unique weak solution $u = u_{\nu,k}$ of

$$\begin{cases} \mathcal{L}_\mu u + g_k(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

Proposition

Let $\nu \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$. Then the sequence of weak solutions $\{u_{\nu,k}\}$ of

$$\begin{cases} \mathcal{L}_\mu u + g_k(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.31}$$

decreases and converges, when $k \to \infty$, to some nonnegative function u and there exists a measure $\nu^* \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ such that $0 \leq \nu^* \leq \nu$ and $u = u_{\nu^*}$.

The proof is similar to Proposition 4.1 in Bidaut-Véron and L. Véron, *Inventiones Math. (1991).*

- Let $\nu, \nu' \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$. If $\nu' \le \nu$ and $\nu = \nu^*$, then $\nu' = \nu'^*$

- Assume that $\nu = \nu\lfloor_{\Omega^*} + k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$, then $\nu^* = \nu^*\lfloor_{\Omega^*} + k^*\delta_0 \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$ with $\nu^*\lfloor_{\Omega^*} \le \nu\lfloor_{\Omega^*}$ and $k^* \le k$. More precisely,

(i) If $\mu > \mu_0$ and g satisfies (3.11), then $k = k^*$.

(ii) If $\mu = \mu_0$ and g satisfies (3.13), then $k = k^*$.

(ii) If $\mu > \mu_0$ (resp. $\mu = \mu_0$) and g does not satisfy (3.20) (resp. (3.13)), then $k^* = 0$.

- If $\nu \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$, then ν^* is the largest g-good measure smaller or equal to ν.

Outline

- For $\mu \geq \mu_1 := -\frac{N^2}{4}$,

$$\int_{\mathbb{R}^N_+} |\nabla\zeta|^2 + \mu_1 \int_{\mathbb{R}^N_+} \frac{\zeta^2}{|x|^2} dx \geq 0 \quad \text{for all } \zeta \in C_0^\infty(\mathbb{R}^N_+). \tag{4.1}$$

- $\mathcal{L}_\mu$-harmonic functions vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$,

$$\gamma_\mu(r,\sigma) = r^{\alpha_+}\psi_1(\sigma) \quad \text{and} \quad \phi_\mu(r,\sigma) = \begin{cases} r^{\alpha_-}\psi_1(\sigma) & \text{if } \mu > \mu_1, \\ r^{-\frac{N-2}{2}} \ln(r^{-1})\psi_1(\sigma) & \text{if } \mu = \mu_1, \end{cases} \tag{4.2}$$

where $\psi_1(\sigma) = \frac{x_N}{|x|}$ generates $\ker(-\Delta' + (N-1)I)$ in $H_0^1(\mathbb{S}^{N-1}_+)$, and where

$$\alpha_+ := \alpha_+(\mu) = \frac{2-N}{2} + \sqrt{\mu + N^2/4} \quad \text{and} \quad \alpha_- := \alpha_-(\mu) = \frac{2-N}{2} - \sqrt{\mu + N^2/4}. \tag{4.3}$$

- Put $d\gamma_\mu(x) = \gamma_\mu(x)dx$. We define the γ_μ-dual operator $\mathcal{L}^*_\mu$ of $\mathcal{L}_\mu$ by

$$\mathcal{L}^*_\mu\zeta = -\Delta\zeta - \frac{2}{\gamma_\mu}\langle\nabla\gamma_\mu, \nabla\zeta\rangle \quad \text{for all } \zeta \in C^2(\overline{\mathbb{R}}^N_+), \tag{4.4}$$

and we prove that ϕ_μ is, in some sense, the fundamental solution of

$$\mathcal{L}_\mu u = 0 \ \text{ in } \mathbb{R}^N_+, \quad u = \delta_0 \ \text{ on } \partial\mathbb{R}^N_+$$

in the sense that

$$\int_{\mathbb{R}^N_+} \phi_\mu \mathcal{L}^*_\mu \zeta d\gamma_\mu(x) = b_\mu\zeta(0) \quad \text{for all } \zeta \in C_c(\overline{\mathbb{R}^N_+}) \cap C^{1,1}(\mathbb{R}^N_+)$$

• Brezis-Vazquez, *Rev. Mat. Complut.* 1997

In a bounded domain Ω, satisfying the condition

$$(\mathcal{C}\text{-}1) \qquad 0 \in \partial\Omega\,,\ \ \Omega \subset \mathbb{R}^N_+ \ \text{ and } \ \langle x, \mathbf{n}\rangle = O(|x|^2) \ \text{ for all } \ x \in \partial\Omega,$$

Hardy inequality

$$\int_\Omega |\nabla\zeta|^2 + \mu_1 \int_\Omega \frac{\zeta^2}{|x|^2}dx \geq \frac{1}{4}\int_\Omega \frac{\zeta^2}{|x|^2 \ln^2(|x|R_\Omega^{-1})}dx \quad \text{for all } \zeta \in C_c^\infty(\Omega), \qquad (4.5)$$

• Let

$$\ell_\mu^\Omega := \inf\left\{\int_\Omega \left(|\nabla v|^2 + \frac{\mu}{|x|^2}v^2\right)dx : v \in C_c^1(\Omega), \int_\Omega v^2 dx = 1\right\} > 0.$$

This first eigenvalue is achieved in $H_0^1(\Omega)$ if $\mu > \mu_1$, or in the space $H(\Omega)$ which is the closure of $C_c^1(\Omega)$ for the norm

$$v \mapsto \|v\|_{H(\Omega)} := \sqrt{\int_\Omega \left(|\nabla v|^2 + \frac{\mu_1}{|x|^2}v^2\right)dx},$$

when $\mu = \mu_1$. We set

$$H_\mu(\Omega) = \begin{cases} H_0^1(\Omega) & \text{if } \mu > \mu_1, \\ H(\Omega) & \text{if } \mu = \mu_1. \end{cases}$$

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• Under the assumption $(\mathcal{C}\text{-}1)$ the imbedding of $H_\mu(\Omega)$ in $L^2(\Omega)$ is compact. We denote by γ_μ^Ω the positive eigenfunction, its satisfies

$$\mathcal{L}_\mu \gamma_\mu^\Omega = \ell_\mu^\Omega \gamma_\mu^\Omega \;\text{ in }\; \Omega, \quad \gamma_\mu^\Omega = 0 \;\text{ on }\; \partial\Omega \setminus \{0\}. \tag{4.6}$$

• there exist $c_j = c_j(\Omega, \mu) > 0$, $j = 1, 2$, such that

$$\begin{array}{lll} (i) & \gamma_\mu^\Omega(x) = c_1 \rho(x)|x|^{\alpha_+ - 1}(1 + o(1)) & \text{as } x \to 0, \\ (ii) & |\nabla \gamma_\mu^\Omega(x)| \le c_2 \gamma_\mu^\Omega(x)/\rho(x) & \text{for all } x \in \Omega. \end{array} \tag{4.7}$$

• We first characterize the positive $\mathcal{L}_\mu$-harmonic functions which are singular at 0.

Theorem

Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ and $\mu \ge \mu_1$. If u is a nonnegative $\mathcal{L}_\mu$-harmonic function in Ω vanishing on $B_{r_0}(0) \cap (\partial\Omega \setminus \{0\})$ for some $r_0 > 0$, then there exists $k \ge 0$ such that

$$\lim_{x \to 0} \frac{u(x)}{\rho(x)|x|^{\alpha_- - 1}} = k \quad \text{if } \mu > \mu_1$$

and

$$\lim_{x \to 0} \frac{|x|^{\frac{N}{2}} u(x)}{\rho(x) \ln |x|} = -k \quad \text{if } \mu = \mu_1.$$

- Existence:

Theorem

Let Ω be a C^2 bounded domain satisfying $(\mathcal{C}\text{-}1)$ and $\mu \geq \mu_1$. Then there exists a positive $\mathcal{L}_\mu$-harmonic function in Ω, which vanishes on $\partial\Omega \setminus \{0\}$, which satisfies

$$\phi_\mu^\Omega(x) = \rho(x)|x|^{\alpha_- -1}(1 + o(1)) \quad \textit{as} \;\; x \to 0, \tag{4.8}$$

if $\mu > \mu_1$, and

$$\phi_{\mu_1}^\Omega(x) = \rho(x)|x|^{-\frac{N}{2}}(|\ln|x|| + 1)(1 + o(1)) \quad \textit{as} \;\; x \to 0, \tag{4.9}$$

if $\mu = \mu_1$.

- ϕ_μ^Ω is the unique function belonging to $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$, which satisfies

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = k c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega), \tag{4.10}$$

where $d\gamma_\mu^\Omega = \gamma_\mu^\Omega dx$, here and in the sequel the test function space

$$\mathbb{X}_\mu(\Omega) = \left\{ \zeta \in C(\overline{\Omega}) : \; \gamma_\mu^\Omega \zeta \in H_\mu(\Omega) \text{ and } \; \rho \mathcal{L}_\mu^* \zeta \in L^\infty(\Omega) \right\}.$$

Furthermore, if u is a nonnegative $\mathcal{L}_\mu$-harmonic function vanishing on $\partial\Omega \setminus \{0\}$, there exists $k \geq 0$ such that $u = k\phi_\mu^\Omega$.

- Denote by $\mathfrak{M}(\Omega;\gamma_\mu^\Omega)$ the set of Radon measures ν in Ω such that

$$\sup\left\{\int_\Omega \zeta d|\lambda| : \zeta\in C_c(\Omega),\, 0\le\zeta\le\gamma_\mu^\Omega\right\} := \int_\Omega \gamma_\mu^\Omega d|\nu| < +\infty.$$

If $\nu\in\mathfrak{M}_+(\Omega;\gamma_\mu^\Omega)$ the measure $\gamma_\mu^\Omega\nu$ is a nonnegative bounded measure in Ω. Put

$$\beta_\mu^\Omega(x) = -\frac{\partial\gamma_\mu^\Omega(x)}{\partial\mathbf{n_x}} = \lim_{t\to0^+}\frac{\gamma_\mu^\Omega(x-tn_x)}{t} = \lim_{t\to0^+}\frac{\gamma_\mu^\Omega(x-tn_x)}{\rho^*(x-tn_x))},\quad \forall\, x\in\partial\Omega\setminus\{0\} \tag{4.11}$$

and then

$$c_1|x|^{\alpha_+-1}\le\beta_\mu^\Omega(x)\le c_1c_3|x|^{\alpha_+-1}\quad\text{for } x\in\partial\Omega\setminus\{0\}. \tag{4.12}$$

- Denote

$$\beta_\mu(x)=|x|^{\alpha_+-1}\quad\text{for } x\in\mathbb{R}^N\setminus\{0\}. \tag{4.13}$$

Denote $\mathfrak{M}(\partial\Omega\setminus\{0\};\beta_\mu)$ the set Radon measures λ in $\partial\Omega\setminus\{0\}$ such that

$$\sup\left\{\int_{\partial\Omega\setminus\{0\}}\zeta d|\lambda| : \zeta\in C_c(\partial\Omega\setminus\{0\}),\, 0\le\zeta\le\beta_\mu\right\} := \int_{\partial\Omega\setminus\{0\}}\beta_\mu d|\lambda| < +\infty.$$

- the existence and uniqueness of a solution to

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = \lambda + k\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

Theorem

Let Ω be a C^2 bounded domain satisfying $(\mathcal{C}\text{-}1)$ and $\mu \geq \mu_1$. If $\nu \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega)$, $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $k \in \mathbb{R}$, the function

$$u = \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega := \mathbb{H}_\mu^\Omega[(\nu, \lambda, k)] \tag{4.15}$$

is the unique solution of (4.14) in the very weak sense that $u \in L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$ and

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) + k c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega). \tag{4.16}$$

Theorem

Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ satisfying $(\mathcal{C}\text{-}1)$, $\mu \geq \mu_1$ and u be a nonnegative $\mathcal{L}_\mu$-harmonic functions in Ω. Then there exist $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $k \geq 0$, such that

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega = \mathbb{H}_\mu^\Omega[(0, \lambda, k)].$$

The couple $(\lambda, k\delta_0)$ is called the boundary trace of u.

Thank you!

Happy birthday!!!