Large solutions for some nonlinear equations with Hardy potential.

Moshe Marcus

Department of Mathematics, Technion
E-mail: marcusm@math.technion.ac.il

Conference in honor of Marie-Francoise Bidaut-Véron and Laurent Véron on the occasion of their 70th birthday

The problem

$$-L_\mu u + f(u) = 0 \quad \text{in} \quad \Omega, \quad L_\mu := (\Delta + \frac{\mu}{\delta^2}), \qquad \text{(Eq1)}$$

$\mu \in \mathbb{R}$, Ω a bounded domain in $\mathbb{R}^N$

$$\delta(x) = \operatorname{dist}(x, \partial\Omega),$$

The nonlinear term is an *absorption term*: positive, monotone increasing, superlinear.

We shall discuss the question of existence and uniqueness of *large solutions* of (Eq1) for arbitrary $\mu > 0$.

Some previous works on large solutions of (Eq1)

• Bandle, Moroz and Reichel (2008) studied positive solutions of (Eq1), in smooth domains, in the case $f(u) = u^p$, $p > 1$. They established:

(i) An extension of the Keller – Osserman inequality and

(ii) the existence of large solutions when

either $0 \leq \mu \leq 1/4$,

or $\mu < 0$, $p > 1 - \frac{2}{\alpha_-}$, $\alpha_- := \frac{1}{2} - (\frac{1}{4} - \mu)^{1/2}$.

Some previous works on large solutions of (Eq1)

• Bandle, Moroz and Reichel (2008) studied positive solutions of (Eq1), in smooth domains, in the case $f(u) = u^p$, $p > 1$. They established:

(i) An extension of the Keller – Osserman inequality and

(ii) the existence of large solutions when

either $0 \le \mu \le 1/4$,

or $\mu < 0$, $p > 1 - \frac{2}{\alpha_-}$, $\alpha_- := \frac{1}{2} - (\frac{1}{4} - \mu)^{1/2}$.

• Du and Wei (2015) studied the same problem and proved existence and uniqueness of large solutions for *arbitrary* $\mu > 0$.

Some previous works on large solutions of (Eq1)

• Bandle, Moroz and Reichel (2008) studied positive solutions of (Eq1), in smooth domains, in the case $f(u) = u^p$, $p > 1$. They established:

(i) An extension of the Keller – Osserman inequality and

(ii) the existence of large solutions when
either $0 \leq \mu \leq 1/4$,
or $\mu < 0$, $p > 1 - \frac{2}{\alpha_-}$, $\alpha_- := \frac{1}{2} - (\frac{1}{4} - \mu)^{1/2}$.

• Du and Wei (2015) studied the same problem and proved existence and uniqueness of large solutions for *arbitrary* $\mu > 0$.

• Some related questions, for $f(u) = u^p$, including the case $0 < p < 1$, have been studied by Bandle and Pozio (2019).

• Bandle, Moroz and Reichel (2010) studied equation (Eq1), in smooth domains, in the case $f(u) = e^u$. (In this case large solutions may become negative away from $\partial\Omega$.) They showed that, for $0 < \mu < c_H(\Omega)$ (= Hardy constant in Ω) there exists a unique large solution.

• Positive solutions of equation

$$-\Delta u - \frac{\mu}{|x|^2} u + u^p = 0, \quad x \in \Omega \setminus 0, \quad \mu \leq (N-2)^2/4$$

including the behavior of large solutions, have been studied by Guerch and Veron (1991), Cirstea (2014), Du and Wei (2017) a.o. The latter investigated the case $\mu > (N-2)^2/4$.

• More recently several papers dealt with b.v.p.'s for (Eq1), $f(u) = u^p$:

M and P.T. Nguyen (2014, 2017, 2019), Gkikas and Veron (2015), Gkikas and P.T. Nguyen (2019) a.o.

Conditions on f

$$f \in C^1[0,1), \quad f(0) = 0, \\ f' > 0 \text{ and } f \text{ convex on } (0,\infty). \tag{F1}$$

$\exists a^{\#} > 0$ such that,

$$h(u(x)) \leq a^{\#}\delta(x)^{-2} \quad \forall x \in \Omega,. \tag{F2}$$

for every positive solution u of

$$-\Delta u + f(u) = 0. \tag{Eq0}$$

These conditions hold for a large family of functions including

$$f(u) = u^p,\ p > 1, \qquad f(u) = e^u - 1.$$

Some useful facts

I. Condition (F2) implies the Keller – Osserman condition:

$$\psi(a) = \int_a^\infty \frac{ds}{\sqrt{2F(s)}} < \infty \quad \forall a > 0, \tag{KO}$$

where $F(s) = \int_0^s f(t)dt$.

Some useful facts

I. Condition (F2) implies the Keller – Osserman condition:

$$\psi(a) = \int_a^\infty \frac{ds}{\sqrt{2F(s)}} < \infty \quad \forall a > 0, \tag{KO}$$

where $F(s) = \int_0^s f(t)dt$.

II. Condition (F1) implies:

If Ω is a bounded Lipschitz domain then, equation (Eq0) possesses a unique large solution. (M and Veron, 2006). We denote this solution by U_f^Ω.

III. Conditions (F1), (F2) imply:

If Ω is smooth then,

$$\lim_{\delta\to 0}\frac{U_f^{\Omega}}{\phi}=1, \qquad \phi:=\psi^{-1},$$

(Bandle and M 1992, 1998)

IV. Condition (F2) for (Eq0) implies that a similar inequality holds for (Eq1):

$\exists a_1,\ a_0>0:$

$$h(u/a_1)\leq a_0\delta^{-2}\quad \text{in } \Omega, \tag{F2'}$$

for every positive solution of (Eq1). (M and P.T. Nguyen 2018)

V. Condition (F1) implies that the function $h(t):=f(t)/t$, $t>0$, is monotone increasing.

Main results

Theorem (A)

Let Ω be a bounded Lipschitz domain. Assume that f satisfies (F1) and (F2).
Then for every $\mu \geq 0$, there exists a large solution U of (Eq1) such that $U > U_f^\Omega$.

Theorem (B)

Let Ω be a bounded C^2 domain. Assume that f satisfies (F1) and (F2).
If $0 \leq \mu < 1/4$ then (Eq1) has a unique large solution.

Theorem (C)

Let Ω be a bounded C^2 domain. Assume that f satisfies (F1), (F2). In addition assume:

For every $a > 1$ there exist $\alpha > 1$ and $t_0 > 0$ such that

$$ah(t) \leq h(\alpha t), \quad t > t_0. \tag{1}$$

For every $b \in (0, 1)$ there exist $\beta > 0$ and $t_0 > 0$ such that

$$h(\beta t) \leq bh(t), \quad t > t_0. \tag{2}$$

Finally assume that, $\exists A > 1$ such that

$$h(\phi) \leq A\delta^{-2} \tag{3}$$

Then, for every $\mu > 0$, (Eq1) has a unique large solution in Ω.

The above assumptions are satisfied by, among others, superlinear powers $f(u) = u^p$, $p > 1$ and exponentials e.g. $f(u) = e^u - 1$, $f(u) = u\, e^u$.

Recall that Du and Wei (2015) proved that (Eq1) has a unique large solution for every $\mu > 0$ *in the case of powers* and their proof was strongly dependent on special features of this case.
Theorem C - which applies to a much larger family of nonlinearities - is based on an entirely different approach.

The result of Bandle, Moroz and Reichel (2010) for $f(u) = e^u$, may be compared to our Theorem B. In the latter we consider $f(u) = e^u - 1$ (in which case large solutions are positive everywhere) and allow $0 < \mu < 1/4$ rather then the more restrictive $0 < \mu < c_H$. Surprisingly, there is a difference in the behavior of the large solution near the boundary.

If $f(u) = e^u$, $0 < \mu < c_H$ the large solution V behaves as follows:

$$V(x) \sim \log \delta(x)^{-2} \quad \text{as } \delta(x) \to 0,$$

i.e.

$$h(V(x)) \sim \delta(x)^{-2} / \log \delta(x).$$

If $f(u) = e^u - 1$, $0 < \mu < 1/4$ the large solution U fluctuates between the bounds,

$$c_2\, \delta(x)^{-2} / \log \delta(x) \leq h(U(x)) \leq c_1\, \delta(x)^{-2}$$

and the upper bound is achieved for arbitrarily small δ.

On proof of Thm. A

The existence result is a consequence of the fact that U_f^Ω is a subsolution of (Eq1) and condition (F2).

If $\{\Omega_n\}$ is an exhaustion of Ω and u_n satisfies

$$-\Delta u - \frac{\mu}{\delta^2} u + f(u) = 0 \quad \text{in } \Omega_n,$$

$$u = U_f^\Omega \quad \text{on } \partial\Omega_n$$

then $U_f^\Omega < u_n$ and $\{u_n\}$ increases. In addition by (F2) -or (F2')- $\{u_n\}$ is uniformly bounded in compact subsets of Ω.
Thus $U = \lim u_n$ is a large solution of (Eq1).

On proof of Thm. B

Let u_1, u_2 be two large solutions. We may assume $u_1 \leq u_2$. We show that,

$$\limsup_{x \to \partial\Omega} \frac{u_2}{u_1} \leq 1$$

and therefore $u_1 = u_2$.

The main step is a construction that is reminiscent of one used in M and Veron (1997) – to prove uniqueness for (Eq0) – but is essentially different in the present case.

Notation: Let $P \in \partial\Omega$ and let $\xi = \xi_P$ be an orthogonal set of coordinates with origin at P and ξ_1-axis in the direction of $\mathbf{n}_P$ (quasi-normal into the domain). Let T_P be a cylinder with axis along the ξ_1 axis:

$$T_P = \{\xi = (\xi_1, \xi') : |\xi_1| < \rho,\ |\xi'| < r\}.$$

Since Ω is Lipschitz $\exists\ \rho, k$ such that, for every $P \in \partial\Omega$:

$\exists\ F_P \in Lip(\mathbb{R}^{N-1})$ with Lip constant k s.t. $F_P(0) = 0$ and

$$Q_P := T_P \cap \Omega = \{\xi : F_P(\xi') < \xi_1 < \rho,\ |\xi'| < r = \rho/10k\}.$$

We construct a subsolution w of (Eq1) in

$$Q' := \{\xi : F_P(\xi') < \xi_1 < \rho/2,\ |\xi'| < r/2\},$$

such that:

$$w \in C(\bar{Q}' \cap \Omega), \quad w = 0 \quad \text{on } \partial Q' \cap \Omega,$$

$$\frac{w}{u_2} \to 1 \quad \text{as } \xi \to \partial Q' \cap \partial\Omega.$$

We construct a subsolution w of (Eq1) in

$$Q' := \{\xi : F_P(\xi') < \xi_1 < \rho/2,\ |\xi'| < r/2\},$$

such that:

$$w \in C(\bar{Q}' \cap \Omega), \quad w = 0 \quad \text{on } \partial Q' \cap \Omega,$$
$$\frac{w}{u_2} \to 1 \quad \text{as } \xi \to \partial Q' \cap \partial\Omega.$$

The construction is based on two facts:

(a) L_μ has a Green function in Ω_ρ and
(b) The *boundary Harnack principle* applies to L_μ.

These follow from the assumptions that Ω is Lipschitz and that $0 \leq \mu < 1/4$.

Note that we do not require $\mu < c_H(\Omega)$. As the construction is within a thin boundary strip, $\mu < \frac{1}{4}$ is sufficient (M + Mizel + Pinchover (1998)).

Let $\beta \in (0, \rho/10)$, denote:

$$w_\beta(\xi) := w(\xi_1 + \beta, \xi') \quad \forall x \in Q'_\beta,$$

$$Q'_\beta := \{\xi : F_P(\xi') < \xi_1 < \frac{1}{2}\rho - \beta,\ |\xi'| < \frac{1}{2} r\}.$$

Then, as $\mu > 0$,

$$-L_\mu w_\beta + f(w_\beta) < 0 \quad \text{in } Q'_\beta$$

$$w_\beta < u_1 \quad \text{on } \partial Q'_\beta.$$

Hence: $\quad w_\beta < u_1, \quad$ in $\ Q'_\beta$ which implies

$$w \leq u_1 \quad \text{in } \ Q'.$$

Since $\dfrac{w}{u_2} \to 1$ as $\xi \to \partial Q' \cap \partial\Omega$:

$$\limsup_{\xi_1 \to F_P(\xi'),\ |\xi'| < r/4} u_2/u_1 \leq 1.$$

On proof of Thm. C

We describe main steps.

• Exists maximal solution U^{Ω}_{max} of (Eq1).

$$U^{\Omega}_{f} < U^{\Omega}_{max}, \qquad U^{\Omega}_{max} \leq A\tilde{\phi}, \qquad \tilde{\phi} = h^{-1}(\delta^{-2}).$$

• If U is a large solution then:

$$\mu \leq \limsup_{x \to \partial\Omega} h(U)\delta^{2}.$$

• If Ω is a ball $B_R(0)$ or the exterior of a ball $B'_R(0) = \mathbb{R}^N \setminus B_R(0)$ then,

(i) U^{Ω}_{max} is a r.s. large solution, unique in the class of r.s. solutions.

(ii) $\exists\ \delta_n \downarrow 0\ :\quad \dfrac{\mu}{\delta_n^2} \le h(U^{\Omega}_{max}(x)),$

$$|x| = R - \delta_n \ \text{ if }\ \Omega = B_R(0), \qquad |x| = R + \delta_n \ \text{ if }\ \Omega = B'_R(0).$$

The sequence $\{\delta_n\}$ may depend on R.

• Given $P \in \partial\Omega$, let B_R^P be a ball such that $\bar{B}_R^P \cap \bar{\Omega} = \{P\}$. Let U_R^P be the r.s. large solution of (Eq1) in $\mathbb{R}^N \setminus B_R^P$.

• Since $\Omega \subset \mathbb{R}^N \setminus B_R^P$,

U_R^P is a subsolution of (Eq1) in Ω.

Therefore, if U large solution of (Eq1) in Ω then

$$U_R^P \leq U, \quad \text{in } \Omega.$$

• Given $P \in \partial\Omega$, let B_R^P be a ball such that $\bar{B}_R^P \cap \bar{\Omega} = \{P\}$. Let U_R^P be the r.s. large solution of (Eq1) in $\mathbb{R}^N \setminus B_R^P$.

• Since $\Omega \subset \mathbb{R}^N \setminus B_R^P$,

U_R^P is a subsolution of (Eq1) in Ω.

Therefore, if U large solution of (Eq1) in Ω then

$$U_R^P \le U, \quad \text{in } \Omega.$$

• Since Ω is smooth, we may choose R independent of P. Then,

$$W := \max_{P \in \partial\Omega} U_R^P \le U, \quad \text{in } \Omega. \qquad (*)$$

W is a subsolution of (Eq1) in Ω. The smallest solution dominating W is a large solution. By (*) it is the smallest large solution of (Eq1). Denote it: U_{min}^{Ω}.

- There exists sequence $\delta_n \downarrow 0$ such that,

$$\mu h(\tilde{\phi}(\delta_n)) = \frac{\mu}{\delta_n^2} \leq h(W) \leq h(U_{min}^{\Omega}(x))$$

on $\Gamma_n := [x \in \Omega : \delta(x) = \delta_n], \quad n = 1, 2, \ldots.$

- There exists sequence $\delta_n \downarrow 0$ such that,

$$\mu h(\tilde{\phi}(\delta_n)) = \frac{\mu}{\delta_n^2} \leq h(W) \leq h(U^{\Omega}_{min}(x))$$

on $\Gamma_n := [x \in \Omega : \delta(x) = \delta_n], \quad n = 1, 2, \dots.$

- $\exists\, M > 1$ s.t. $U^{\Omega}_{max} \leq M\, U^{\Omega}_{min}$ on Γ_n. Hence,

$$U^{\Omega}_{max} \leq M\, U^{\Omega}_{min} \quad \text{in } \Omega.$$

By an argument based on convexity of f (M+Veron 1998),

$$U^{\Omega}_{max} = U^{\Omega}_{min}.$$

□

THANK YOU FOR YOUR ATTENTION.