Weighted and pointwise bounds in measure datum problems with applications

Nguyen Cong Phuc
Louisiana State University, USA

LSU

ShanghaiTech University and Masaryk University
Zoom Talk – June 1st, 2020

In celebration of
Marie-Françoise Bidaut-Véron and Laurent Véron's 70th birthday

Acknowledgments

- **Quoc-Hung Nguyen, Shanghai Tech University**
- **Simons Foundation**

Gradient estimates – Introduction

Consider the equation

$$-\Delta u = f \qquad \text{in} \quad \mathbb{R}^n.$$

Then, under mild assumptions on f and u, one has a pointwise representation

$$u(x) = c(n) \int_{\mathbb{R}^n} \Gamma(x-y) f(y) dy,$$

where

$$\Gamma(x-y) = \begin{cases} |x-y|^{2-n} & \text{if} \quad n > 2 \\ -\log(|x-y|) & \text{if} \quad n = 2. \end{cases}$$

Gradient estimates – Introduction

This pointwise representation is often written as

$$u(x) = \mathbf{I}_2 f(x), \qquad n > 2,$$

and by differentiating

$$|\nabla u(x)| \leq c\, \mathbf{I}_1 |f|(x), \qquad n \geq 2.$$

Here $\mathbf{I}_\alpha$, $\alpha \in (0, n)$ is a fractional integral

$$\mathbf{I}_\alpha \mu(x) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{n-\alpha}}$$

$$= c \int_0^\infty \frac{\mu(B_t(x))}{t^{n-\alpha}} \frac{dt}{t}.$$

Gradient estimates – Introduction

Now recall the fractional maximal function $\mathbf{M}_\alpha, \alpha \in (0, n)$:

$$\mathbf{M}_\alpha \mu(x) := \sup_{t>0} \frac{\mu(B_t(x))}{t^{n-\alpha}}, \quad x \in \mathbb{R}^n.$$

Obviously, one has

$$\mathbf{M}_\alpha \mu \leq c\, \mathbf{I}_\alpha \mu \quad \text{in } \mathbb{R}^n.$$

The converse holds in the following sense:

Theorem (Muckenhoupt-Wheeden '74)

Let $q > 0$ and w be a weight in the $\mathbf{A}_\infty$ class. We have

$$\int_{\mathbb{R}^n} (\mathbf{I}_\alpha \mu)^q w dx \leq C(q, n, [w]_{A_\infty}) \int_{\mathbb{R}^n} (\mathbf{M}_\alpha \mu)^q w dx.$$

Gradient estimates – Introduction

We recall that $w \in \mathbf{A}_\infty$ **if there exist** $C, \nu > 0$ **such that**

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|}\right)^{\nu},$$

for all balls B **and all measurable set** $E \subset B$**. The pair** (C, ν) **is called the** $\mathbf{A}_\infty$ **constants of** w **and is denoted by** $[w]_{\mathbf{A}_\infty}$**.**

Gradient estimates – Introduction

• **Thus for the solution u above one has**

$$\int_{\mathbb{R}^n} |\nabla u|^q w dx \leq C(q, n, [w]_{A_\infty}) \int_{\mathbb{R}^n} (\mathbf{M}_1|f|)^q w dx$$

for all weights $w \in \mathbf{A}_\infty$.

• **This bound has the advantage that it could hold for more general linear uniformly elliptic operator with possibly discontinuous coefficients, whereas for the pointwise bound $|\nabla u| \leq c\mathbf{I}_1|f|$ to hold one needs at least Dini continuous coefficients.**

• **Easier to handle up to the boundary of a bounded domain.**

• **Moreover, this bound is usually enough in many applications to nonlinear PDEs.**

Main goals

• **To obtain Muckenhoupt-Wheeden type (weighted) bounds for gradients of solutions to quasilinear elliptic equations with measure data:**

$$\begin{cases} -\Delta_p u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and μ is a finite signed Radon measure in Ω.

• **To obtain pointwise estimates for gradients of solutions to**

$$-\Delta_p u = \mu \quad \text{in } \Omega.$$

Assumption on p: For pointwise gradient estimate, we will be dealing mainly with the case

$$\frac{3n-2}{2n-1} < p < +\infty.$$

Main goals

• **As an application, we obtain characterizations of existence and removable singularities for the quasilinear Riccati type (viscous Hamilton-Jacobi type) equation with measure data:**

$$\begin{cases} -\Delta_p u = |\nabla u|^q + \mu & \textbf{in } \Omega, \\ u = 0 & \textbf{on } \partial\Omega. \end{cases} \tag{2}$$

Remark: We can deal with all $p > 1$ and $q \geq 1$ for this equation.

A remark on principal operator

We can replace the p-Laplacian $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with a more general operator of the form

$$\mathcal{L}_p(u) = \mathrm{div} A(x, \nabla u),$$

where the nonlinearity $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain ellipticity and regularity conditions.

• For (integral) Muckenhoupt-Weeden type bounds, we need that A satisfies a VMO condition in the x-variable.

• For pointwise gradient bounds, we need that A satisfies a Hölder or Dini condition in the x-variable.

Assumptions on Ω

For the global gradient estimates, we also require certain regularity on the ground domain Ω. For our purpose C^1 domains would be enough. A sharper condition on $\partial\Omega$ is the so-called Reifenberg flatness condition. Namely, at each boundary point and at every scale, we ask that the boundary of Ω be trapped between two hyperplanes separated by a distance proportional to the scale.

Muckenhoupt-Wheeden type (weighted) bounds

Theorem (P., Adv. Math. '14; Nguyen-P., Math. Ann. '19)

Let $\mu \in M_b(\Omega)$. *Let* $\frac{3n-2}{2n-1} < p < \infty$ *and* $q > 0$. *For any* $w \in \mathbf{A}_\infty$ *and any renormalized solution* u *to*

$$-\Delta_p u = \mu \quad in \quad \Omega, \qquad u = 0 \quad on \quad \partial\Omega,$$

we have

$$\int_\Omega |\nabla u|^q w(x) dx \leq C \int_\Omega \mathbf{M}_1(\mu)^{\frac{q}{p-1}} w(x) dx.$$

Here C *depends only on* $n, p, q, [w]_{\mathbf{A}_\infty}$, *and* Ω.

• **Nguyen-P. (submitted): A similar bound is obtained for the case** $1 < p \leq \frac{3n-2}{2n-1}$**, but with** $q > 2 - p$ **and** $w \in \mathbf{A}_{\frac{q}{2-p}}$**.**
• **Local unweighted setting: Mingione, Math. Ann. '10. This paper treats with measurable coefficients for** $q \leq p + \epsilon$**.**

The key comparison estimate

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution of $-\Delta_p u = \mu$. For $B_R = B_R(x_0) \Subset \Omega$, we let $w \in W^{1,p}_0(B_R) + u$ be the unique solution to the equation

$$\begin{cases} -\Delta_p w &= 0 \quad \text{in} \quad B_R, \\ w &= u \quad \text{on} \quad \partial B_R. \end{cases}$$

Lemma

Assume that $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. *Then for any* $\frac{n}{2n-1} < \gamma_0 < \frac{(p-1)n}{n-1} \leq 1$,

$$\left(\fint_{B_R} |\nabla u - \nabla w|^{\gamma_0} dx \right)^{\frac{1}{\gamma_0}} \leq C \left[\frac{|\mu|(B_R)}{R^{n-1}} \right]^{\frac{1}{p-1}} + \\ + C \frac{|\mu|(B_{2R})}{R^{n-1}} \left(\fint_{B_R} |\nabla u|^{\gamma_0} dx \right)^{\frac{2-p}{\gamma_0}}.$$

For $p > 2 - \frac{1}{n}$, this was obtained by Mingione '07, Mingione-Duzaar '11, with $\gamma_0 = 1$. The case $1 < p \leq \frac{3n-2}{2n-1}$ is still open.

A difficulty arises when p gets small

Note that the fundamental solution of the p-Laplace equation is given by

$$v(x) = c(n,p)|x|^{\frac{p-n}{p-1}}, \qquad x \in \mathbb{R}^n.$$

Thus $\nabla v \in L^{\frac{n(p-1)}{n-1},\infty}$, and $\nabla v \in L^1_{\mathrm{loc}}$ if and only if $p > 2 - \frac{1}{n}$. When $p \leq 2 - \frac{1}{n}$, $\nabla v \notin L^1_{\mathrm{loc}}$, and this prevents us from using the Sobolev's inequality in the 'traditional' argument. Note that

$$\frac{3n-2}{2n-1} < 2 - \frac{1}{n}.$$

(However, $|\nabla(v^\delta)| = \delta|\nabla v|v^{\delta-1} \in L^1_{\mathrm{loc}}$ for certain $\delta \in (0,1)$.)

Pointwise gradient estimates by Wolff's potentials

- **Recall the estimates for functions:**

Theorem (Kilpeläinen-Malý, Acta Math. '94)

Suppose that $u \geq 0$ is a solution of $-\Delta_p u = \mu$ in Ω. Then for any ball $B_{2r}(x) \subset \Omega$ and any $\gamma > 0$, we have

$$u(x) \geq c_1 \int_0^r \left[\frac{\mu(B_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

$$u(x) \leq c_2 \int_0^{2r} \left[\frac{\mu(B_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} + c_2 \left(\fint_{B_{2r}(x)} u^\gamma\right)^{\frac{1}{\gamma}}.$$

Pointwise gradient estimates by Wolff's potentials

- **For derivatives:**

Theorem (Duzaar-Mingione JFA '10, Kuusi-Mingione ARMA '13)

Suppose that u is a solution of $-\Delta_p u = \mu$ in Ω, where $p > 2 - \frac{1}{n}$. Then for any ball $B_{2r}(x) \subset \Omega$, we have

$$|\nabla u(x)| \leq C \left[\int_0^{2r} \frac{|\mu|(B_t(x))}{t^{n-1}} \frac{dt}{t} \right]^{\frac{1}{p-1}} + C \fint_{B_{2r}(x)} |\nabla u| dy.$$

- **Note that $\int_0^{2r} \frac{|\mu|(B_t(x))}{t^{n-1}} \frac{dt}{t}$ is a truncated first order (linear) Riesz's potential of $|\mu|$.**
- **Historically, the nonlinear case with $p = 2$ was done in [Mingione JEMS '11]; the case $p > 2$ was done in [Duzaar-Mingione AJM '11].**

Pointwise gradient estimates by Wolff's potentials

Theorem (Nguyen-P. JFA '20)

Suppose that u is a solution of $-\Delta_p u = \mu$ in Ω, where $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. Then for any ball $B_{2r}(x) \subset \Omega$, we have

$$|\nabla u(x)| \leq C \left[\int_0^{2r} \left(\frac{|\mu|(B_t(x))}{t^{n-1}} \right)^{\gamma_0} \frac{dt}{t} \right]^{\frac{1}{\gamma_0(p-1)}} + C \left[\fint_{B_{2r}(x)} |\nabla u|^{\gamma_0} \right]^{\frac{1}{\gamma_0}}.$$

Here γ_0 is any number in $\left(\frac{n}{2n-1}, \frac{(p-1)n}{n-1} \right)$.

- $\gamma_0 < 1$.
- $\int_0^{2r} \left(\frac{|\mu|(B_t(x))}{t^{n-1}} \right)^{\gamma_1} \frac{dt}{t} \leq C \int_0^{4r} \left(\frac{|\mu|(B_t(x))}{t^{n-1}} \right)^{\gamma_2} \frac{dt}{t}$ **whenever** $\gamma_1 > \gamma_2 > 0$.

Sharp quantitative $C^{1,\alpha}$ estimates

Two important ingredients in the proof of the above theorem:

- **Comparision estimates obtained in a previous lemma.**
- **Sharp quantitative $C^{1,\alpha}$ estimates: Let w be a $W^{1,p}_{\rm loc}$ solution to the homogeneous equation $-\Delta_p w = 0$ in Ω. Then we have**

1) Classical $C^{1,\alpha}$ bound:

$$\frac{|\nabla w(x) - \nabla w(y)|}{|x-y|^{\alpha}} \leq C \fint_{B_r(x_0)} |\nabla w| dx$$

for any $x, y \in B_{r/2}(x_0) \subset B_r(x_0) \subset \Omega$.

2) Duzaar-Mingione's $C^{1,\alpha}$ bound:

$$\fint_{B_\rho(x_0)} |\nabla w - (\overline{\nabla w})_{B_\rho(x_0)}| \leq C \left(\frac{\rho}{r}\right)^{\alpha} \fint_{B_r(x_0)} |\nabla w - (\overline{\nabla w})_{B_r(x_0)}|$$

for every $B_r(x_0) \subset \Omega$ and $\rho < r$.

Sharp quantitative $C^{1,\alpha}$ estimates

Theorem (Nguyen-P. JFA '20)

Let $p > 1$ and $q \in (1, p+1)$, and define a nonlinear vector field

$$U_q(\xi) := |\xi|^{q-2}\xi, \qquad \xi \in \mathbb{R}^n.$$

Then there exist constants $C > 1$ and $\alpha \in (0,1]$ such that

$$\fint_{B_\rho(x_0)} |U_q(\nabla w) - (\overline{U_q(\nabla w)})_{B_\rho(x_0)}| \leq C\left(\frac{\rho}{r}\right)^\alpha \fint_{B_r(x_0)} |U_q(\nabla w) - (\overline{U_q(\nabla w)})_{B_r(x_0)}|$$

for every $B_r(x_0) \subset \Omega$ and $\rho < r$.

- **The case $q = p$: Diening-Kaplický-Schwarzacher '12. For $p > 2$, $q = \frac{p+2}{2}$: Duzaar-Mingione '10. We use with $q = 1 + \gamma_0$, $\gamma_0 \in (\frac{n}{2n-1}, \frac{n(p-1)}{n-1})$. [Acerbi-Fusco '89], [Giaquinta-Modica '86].**

Sharp quantitative $C^{1,\alpha}$ estimates

The above theorem implies that nonlinear vector field $U_q(\nabla w)$ is Hölder continuous with order α. But what is more important here is that this Hölder continuity is quantified in a sharp way. That is, both sides of the bound have the same structure which allows us to apply an iteration process to prove pointwise gradient estimates. We emphasize that the weaker version where the integral

$$\fint_{B_r(x_0)} |U_q(\nabla w) - (\overline{U_q(\nabla w)})_{B_r(x_0)}|$$

on the right is replaced by

$$\fint_{B_r(x_0)} |U_q(\nabla w)|$$

is not enough for our purpose.

Global pointwise gradient estimates

Theorem (Nguyen-P. JFA '20)

Let $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$ and Ω be a bounded C^1 domain. Suppose that u is a renormalized solution u to

$$-\Delta_p u = \mu \ \ in \ \ \Omega, \quad u = 0 \ \ on \ \ \partial\Omega.$$

Then for any $0 < \epsilon << 1$ and $x \in \Omega$ we have

$$|\nabla u(x)| \leq C d(x, \partial\Omega)^{-\epsilon} \left[\int_0^{2\mathrm{diam}(\Omega)} \left(\frac{|\mu|(B_t(x))}{t^{n-1}} \right)^{\gamma_0} \frac{dt}{t} \right]^{\frac{1}{\gamma_0(p-1)}}.$$

Here γ_0 is any number in $\left(\frac{n}{2n-1}, \frac{(p-1)n}{n-1}\right)$.

This also holds for $p > 2 - \frac{1}{n}$ in which case we take $\gamma_0 = 1$. For C^2 domains, we can take $\epsilon = 0$ (Banerjee-Nguyen-P. in preparation).

Applications to Riccati type equations

Next we consider the equation

$$\begin{cases} -\Delta_p u &= |\nabla u|^q + \mu \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad\quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

As it turns out, the capacity $\mathrm{Cap}_{1,s}$ with $s = \frac{q}{q-p+1}$, is the intrinsic capacity associated to (3). This is justified by the following results. First recall that for a compact set $K \subset \mathbb{R}^n$,

$$\mathrm{Cap}_{1,s}(K) = \inf\Big\{ \int_{\mathbb{R}^n} (|\nabla\varphi|^s + \varphi^s) dx : \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq \chi_K \Big\}.$$

Necessary condition for existence

Theorem (Hansson-Maz'ya-Verbitsky '99, P. CPDE '10)

Let $\mu \in M_b^+(\Omega)$ be such that $\operatorname{supp}(\mu) \Subset \Omega$. Suppose that $q > p - 1 > 0$. Then there exists $c_1 > 0$ such that if (3) admits a solution then

$$\mu(K) \leq c_1 \operatorname{Cap}_{1, \frac{q}{q-p+1}}(K)$$

for all compact sets $K \subset \Omega$.

Verbitsky's problem (personal comm.); also a problem stated in [Bidaut-Veron–Garcia-Huidobro–Veron, '13].

Sufficient condition for existence

Theorem (Nguyen-P., Math. Ann. '19 and submitted)

Suppose that $\partial\Omega \in C^1$. Let $1 < p \leq 2 - \frac{1}{n}$ and $q \geq 1$. There exists $c_0 = c_0(n, p, q, \Omega) \in (0, 1)$ such that if $\mu \in M_b(\Omega)$ with

$$|\mu|(K) \leq c_0 \,\mathrm{Cap}_{1, \frac{q}{q-p+1}}(K) \quad \forall K \subset \Omega, \tag{4}$$

then there exists a solution $u \in W_0^{1,q}(\Omega)$ to equation (3) such that

$$\int_K |\nabla u|^q \leq C \,\mathrm{Cap}_{1, \frac{q}{q-p+1}}(K) \quad \forall K \subset \Omega.$$

- **The case $p > 2 - \frac{1}{n}$ and $q > p - 1$ was done in [P. 2014].**
- **For $p = 2$: 'classical' result of Hansson-Maz'ya-Verbitsky 1999.**
- **The case $1 < p \leq \frac{3n-2}{2n-1}$ and the case $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$ are treated differently.**

Nature of space of solutions

The solutions u obtained in the above theorem obey that

$$|\nabla u| \in M_q^{1,s}(\mathbb{R}^n), \quad s = \frac{q}{q-p+1},$$

where $M_q^{1,s}(\mathbb{R}^n)$ consists of $f \in L^q_{\rm loc}(\mathbb{R}^n)$ such that the inequality

$$\left(\int_{\mathbb{R}^n} |\varphi|^s |f|^q dx\right)^{\frac{1}{q}} \leq C\|\varphi\|_{W^{1.s}}^{\frac{s}{q}} \tag{5}$$

holds for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. A norm $f \in M_q^{\alpha,s}$ is defined as the least possible constant C in the above inequality. By a capacitary strong type inequality [Maz'ya-Adams-Dahlberg], one has the equivalence:

$$\|f\|_{M_q^{1,s}} \simeq \sup_K \left(\frac{\int_K |f(x)|^q dx}{\mathrm{Cap}_{1,s}(K)}\right)^{1/q}. \tag{6}$$

Nature of space of solutions

Theorem (Ooi-P. '20)

For $q > 1$, we have

$$\|f\|_{M_q^{1,s}} \simeq \sup_w \left(\int_{\mathbb{R}^n} |f(x)|^q w(x) dx \right)^{1/q},$$

where the supremum is taken over all nonnegative $w \in L^1(C) \cap A_1^{\rm loc}$ with $\|w\|_{L^1(C)} \leq 1$ and $[w]_{A_1^{\rm loc}} \leq \bar{\mathbf{c}}(n,s)$ for a constant $\bar{\mathbf{c}}(n,s) \geq 1$.

- $\|w\|_{L^1(C)} := \int_0^\infty \mathrm{Cap}_{1,s}\{x \in \mathbb{R}^n : |w(x)| > t\} dt$.
- **Closely related work [Verbitsky '80].**
- $M_q^{1,s}$ **is dual space. Many characterizations of its pre-dual are also known [Ooi-P. '20], [Kalton-Verbitsky '98].**
- **Similar resutls for** $M_q^{\alpha,s}$, $\alpha \in (0,n)$, $s > 1$, $q > 1$.

Other cases and unsolved problems

• **The above theorem does not cover the 'sublinear' case $p-1<q<1$ with $\frac{3n-2}{2n-1}<p\leq 2-\frac{1}{n}$. This has been settled recently (Nguyen-P. '20) provided $\mathrm{supp}(\mu)\Subset\Omega$. Global pointwise estimates for the gradient were employed in this case.**

• **The case $n(p-1)/(n-1)\leq q<1$ with $1<p\leq\frac{3n-2}{2n-1}$ is still unsolved.**

• **The case $0<q<n(p-1)/(n-1)$ is a subcritical case [Grenon-Murrat-Porretta '13], [Betta-Mercaldo-Murrat-Porzio '03], and many others.**

Other cases and unsolved problems

- **Distributional data:** **Suppose that $q > p-1$ and μ is a general distribution in Ω. If equation (3) admits a solution u such that**

$$\int_K |\nabla u|^q \leq C\, \mathrm{Cap}_{1, \frac{q}{q-p+1}}(K) \quad \forall K \subset \Omega. \tag{7}$$

then one must have that $\mu = \mathrm{div}\vec{F}$ for a vector field $\vec{F}$ such that

$$\int_K |\vec{F}|^{\frac{q}{p-1}} \leq C\, \mathrm{Cap}_{1, \frac{q}{q-p+1}}(K) \quad \forall K \subset \Omega. \tag{8}$$

- **Conversely, for $q \geq p$, there exsits $c_0 > 0$ such that if (8) holds with $C \leq c_0$ then equation (3) admits a solution u such that (7) holds [Mengesha-P. '16, Nguyen-P. '20], [Ferone-Murat '00, '14], [Adimurthi-P. '18], [Jaye-Maz'ya-Verbitsky] (for $p = q$).**

- **For divergence form data, existence under condition (8) is still unsolved in the sub-natural growth case $p-1 < q < p$.**

An example of oscillatory data

[Mengesha-P. '16], [Maz'ya-Verbitsky '02].
Let

$$f(x) = |x|^{-\epsilon-s}\sin(|x|^{-\epsilon}),$$

where $s = q/(q-p+1)$ **and** $\epsilon > 0$ **such that** $\epsilon + s < n$. **Then** $\mu = |f(x)|dx$ **fails to satisfy the capacitary inequality** (4), **but the condition** (8) **can be employed to show that the equation**

$$-\Delta_p u = |\nabla u|^q + \lambda f, \qquad q \geq p,$$

admits a solution $u \in W_0^{1,q}(B_1(0))$ **provided** $|\lambda|$ **is sufficiently small. Observe that**

$$f = \mathrm{div}(\vec{F}) + \textbf{`a good function'},$$

where

$$\vec{F} = \frac{1}{\epsilon} x|x|^{-s}\cos(|x|^{-\epsilon}).$$

Weighted estimates of Caldéron-Zygmund type

The study of $-\Delta_p u = |\nabla u|^q +$ 'a distribution' has motivated the study of

$$\begin{cases} \Delta_p u = \mathrm{div}\vec{F} & \textbf{in } \Omega, \\ u = 0 & \textbf{on } \partial\Omega, \end{cases}$$

where one looks for weighted Caldéron-Zygmund type bounds of the form:

$$\int_\Omega |\nabla u|^q w(x)dx \leq C \int_\Omega |\vec{F}|^{\frac{q}{p-1}} w(x)dx.$$

- **$q > p > 1$, $w \in \mathbf{A}_{q/p}$: [P. '11, Mengesha-P.], [Iwaniec '83], [Peral-Caffarelli '98], [Kinnunen-Zhou '99, '01], [Byun-Wang '07], [Byun-Yao-Zhou '08], and many others.**
- **$q = p$, $w \in \mathbf{A}_1$: [Adimurthi-P. '16] (related to [Iwaniec-Sborden '94, Lewis '93]).**
- **$p - 1 < q < p$: still open (Iwaniec conjecture).**

Removable sets for $-\Delta_p u = |\nabla u|^q$

Theorem (Nguyen-P., Math. Ann. '19)

Let $1 < p \leq 2 - \frac{1}{n}$ and $q \geq 1$. If a compact set $E \subset \Omega$ is a removable set for the equation $-\Delta_p u = |\nabla u|^q$ in Ω, then it must hold that

$$\mathrm{Cap}_{1, \frac{q}{q-p+1}}(E) = 0.$$

- **The case $p > 2 - \frac{1}{n}$ and $q > p - 1$ was done in [P. 2014].**
- **The case $p - 1 < q < 1$ with $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$ follows from the recent work [Nguyen-P. '20].**

Removable sets for $-\Delta_p u = |\nabla u|^q$

The above result is sharp at least in the natural class of p-superharmonic functions in Ω. Precisely, we have

Theorem (P. CPDE '10)

Let $0 < p - 1 < q$. If E is a compact set in Ω with $\mathrm{Cap}_{1,\frac{q}{q-p+1}}(E) = 0$ then any solution u to

$$\begin{cases} u \text{ is } p\text{-superharmonic in } \Omega, \quad (\text{needed only for } q > p), \\ |\nabla u| \in L^q_{\mathrm{loc}}(\Omega \setminus E), \text{ and} \\ -\Delta_p u = |\nabla u|^q \text{ in } \mathcal{D}'(\Omega \setminus E), \end{cases}$$

is also a solution to

$$\begin{cases} u \text{ is } p\text{-superharmonic in } \Omega, \\ |\nabla u| \in L^q_{\mathrm{loc}}(\Omega), \text{ and} \\ -\Delta_p u = |\nabla u|^q \text{ in } \mathcal{D}'(\Omega). \end{cases}$$

THANK YOU FOR YOUR ATTENTION!