Some classes of solutions to quasilinear elliptic equations of p-Laplace type

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Abstract

This talk is concerned with various classes of solutions, including ***BMO***, Sobolev and Morrey space solutions (along with their local counterparts) to quasilinear elliptic equations of the type

$$-\Delta_p u = \sigma\, u^q + \mu, \quad u \geq 0 \quad \text{in } \mathbf{R}^n,$$

where $p > 1$ and $q > 0$. Here Δ_p is the p-Laplacian, and μ, σ are nonnegative functions (or Radon measures). Solutions u are positive p-superharmonic functions in $\mathbf{R}^n$ (or local renormalized solutions). More general operators $\text{div}\mathcal{A}(x, \nabla\cdot)$ in place of Δ_p will be treated.

We will discuss necessary and sufficient conditions for the existence, and pointwise estimates of solutions, along with related weighted norm inequalities. We intend to cover mostly the exponents q above $(q > p - 1)$ and below $(0 < q < p - 1)$ the **natural growth** case.

Based in part on joint work with Nguyen Cong Phuc (Louisiana State University, USA), Dat Tien Cao (Minnesota State University, USA), and Adisak Seesanea (Hokkaido University, Japan).

Publications

1. *BMO solutions to quasilinear elliptic equations*, in preparation (2020) (with Nguyen Cong Phuc)
2. *Quasilinear elliptic equations with sub-natural growth terms and nonlinear potential theory*, ***Rendiconti Lincei*** **30** (2020), 733–758
3. *Wolff's inequality for intrinsic nonlinear potentials and quasilinear elliptic equations*, ***Nonlin. Analysis*** **194** (2020)
4. *Solutions in Lebesgue spaces to nonlinear elliptic equations with sub-natural growth terms*, ***St. Petersburg Math. J.*** **31** (2020), 557-572 (with Adisak Seesanea)
5. *Finite energy solutions to inhomogeneous nonlinear elliptic equations with sub-natural growth terms*, ***Adv. Calc. Var.*** **13** (2020), 53–74 (with Adisak Seesanea)
6. *Nonlinear elliptic equations and intrinsic potentials of Wolff type*, ***J. Funct. Analysis*** **272** (2017) 112–165 (with Dat Tien Cao)

Quasilinear equations

We consider **p-superharmonic** solutions to the equation

$$-\Delta_p u = \sigma u^q + \mu, \quad u \geq 0 \quad \text{in } \mathbf{R}^n, \tag{1}$$

$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the p-Laplace operator, $p > 1$, $q > 0$; $u \in L^q_{loc}(\sigma)$; $\mu, \sigma \in M_+(\mathbf{R}^n)$ (locally finite Radon measures).

We actually treat more general quasilinear operators $\mathrm{div}\mathcal{A}(x, \nabla)$ in place of Δ_p. We use **local renormalized** solutions introduced by Marie-Françoise [Bidaut-Véron '03]. The equivalence to p-superharmonic solutions was shown by [Kilpeläinen–Kuusi–Tuhola-Kujanpää '11].

The case $q = p - 1$ is called the **natural growth** case (Schrödinger equation if $p = 2$). We distinguish between the sub-natural-growth case $0 < q < p - 1$ and the super-natural-growth case $q > p - 1$ $(\mu \neq 0)$. Similar results hold for the fractional Laplace equation $(0 < \alpha < n)$

$$(-\Delta)^{\frac{\alpha}{2}} u = \sigma u^q + \mu, \quad u \geq 0 \quad \text{in } \mathbf{R}^n. \tag{2}$$

Classes of measures

The ***p*-capacity** of a compact set $K \subset \mathbf{R}^n$ is defined by:

$$\mathrm{cap}_p(K) = \inf \left\{ ||\nabla u||^p_{L^p(\mathbf{R}^n)} : \ u \ge 1 \text{ on } K, \quad u \in C_0^\infty(\mathbf{R}^n) \right\}.$$

For the existence of a nontrivial solution u to (1) with $q > 0$, $p > 1$, the measure σ must be **absolutely continuous** w/r to p-capacity:

$$\mathrm{cap}_p(K) = 0 \Longrightarrow \sigma(K) = 0.$$

More precisely, if u is a nontrivial solution to (1), in the case $0 < q \le p-1$ we have [Cao-V. '17] (recall that $u \in L^q_{loc}(\sigma)$)

$$\sigma(K) \le C \ \left[\mathrm{cap}_p(K)\right]^{\frac{q}{p-1}} \left(\int_K u^q d\sigma \right)^{\frac{p-1-q}{p-1}}.$$

In the case $q \ge p-1$, we have

$$\sigma(K) \le C \, \mathrm{cap}_p(K) \, (\min_K u)^{p-1-q}.$$

Classes of measures (continuation)

For $q = p - 1$, we get the important class of **Maz'ya measures**

$$\sigma(K) \le C \,\mathrm{cap}_p(K).$$

Another important class of measures is associated with **Riesz capacities**

$$\mathrm{cap}_{\alpha,r}(E) = \inf \left\{ ||f||^r_{L^r(\mathbf{R}^n)} : \ \mathbf{I}_\alpha f \ge 1 \text{ on } E, \quad f \in L^r_+(\mathbf{R}^n) \right\},$$

for any $E \subset \mathbf{R}^n$. Here $\mathbf{I}_\alpha = (-\Delta)^{-\frac{\alpha}{2}}$ is the Riesz potential of order $0 < \alpha < n$ and $1 < r < \infty$.
Notice that $\mathrm{cap}_{1,p}(K)$ is equivalent to $\mathrm{cap}_p(K)$.
The corresponding class of **Maz'ya measures** (occurs in the case $q > p - 1$ and $d\sigma = dx$, with $\alpha = p$ and $r = \frac{q}{q-p+1}$),

$$\sigma(K) \le C \,\mathrm{cap}_{\alpha,r}(K)$$

characterizes the weighted norm inequality

$$||\mathbf{I}_\alpha f||_{L^r(d\sigma)} \le C \, ||f||_{L^r(dx)}, \quad \forall f \in L^r(dx). \qquad (3)$$

Weighted norm inequalities for Riesz potentials

More general **two-weight inequalities** (with measures μ and σ)

$$||\mathbf{I}_\alpha(f d\mu)||_{L^q(d\sigma)} \le C\,||f||_{L^r(d\mu)}, \quad \forall f \in L^r(d\mu), \tag{4}$$

play a role for fractional Laplace equations (2). Here $q > 0$ and $r > 1$. In the end-point case $r = 1$, we consider the weighted norm inequality

$$||\mathbf{I}_\alpha \nu||_{L^q(d\sigma)} \le C\,||\nu||, \quad \forall \nu \in M_+(\mathbf{R}^n), \tag{5}$$

where $||\nu||$ denotes the total variation of the (finite) measure ν.
We denote by κ the **least constant** C in the estimates of type (4), (5).
Here the **linear** Riesz potential of $d\mu = f d\nu$ is defined by

$$\mathbf{I}_\alpha(f d\nu) = c \int_{\mathbf{R}^n} \frac{f(y)\, d\nu}{|x-y|^{n-\alpha}} = c \int_0^\infty \frac{\mu(B(x,r))}{r^{n-\alpha}} \frac{dr}{r},$$

where $c = c(\alpha, n) > 0$. We write $I_\alpha f$ if $d\nu = dx$; $I_\alpha \nu$ if $f \equiv 1$.

Nonlinear potentials

The **Wolff potential** (more precisely Havin-Maz'ya-Wolff potential) for $\mu \in M_+(\mathbf{R}^n)$ and $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, is defined by

$$\mathcal{W}_{\alpha,p}\mu(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \mathbf{R}^n.$$

In the special case $\alpha = 1$, we use the notation

$$\mathcal{W}_p\mu(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \mathbf{R}^n.$$

Notice that $\mathcal{W}_p\mu \not\equiv \infty$, equivalently $\mathcal{W}_p\mu(x) < \infty$ q.e., iff

$$\int_1^\infty \left(\frac{\mu(B(0,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty. \tag{6}$$

Weighted norm inequalities for nonlinear potentials

In the case $q > p - 1$, the inequality closely related to (1):

$$||\mathcal{W}_p(f d\mu)||_{L^q(d\sigma)} \le \kappa \, ||f||^{\frac{1}{p-1}}_{L^{\frac{q}{p-1}}(d\mu)}, \quad \forall f \in L^{\frac{q}{p-1}}(d\mu).$$

This is necessary for the existence of a nontrivial supersolution $-\Delta_p u \ge u^q d\sigma + \mu$. A necessary and sufficient condition:

$$\mathcal{W}_p[(\mathcal{W}_p\mu)^q d\sigma] \le C \, \mathcal{W}_p\mu < \infty.$$

In the case $0 < q < p - 1$, the weighted norm inequality related to (1):

$$||\mathcal{W}_p\nu||_{L^q(d\sigma)} \le \kappa \, ||\nu||^{\frac{1}{p-1}}, \quad \forall \nu \in M_+(\mathbf{R}^n).$$

Equivalently, $||\phi||_{L^q(d\sigma)} \le \varkappa \, ||\Delta_p \phi||^{\frac{1}{p-1}}_{L^1(\mathbf{R}^n)}$ for all p-superharmonic test functions ϕ, smooth and vanishing at ∞ in $\mathbf{R}^n$. This inequality holds iff there exists a nontrivial supersolution $u \in L^q(\sigma)$ to $-\Delta_p u \ge \sigma u^q$.

Localized weighted norm inequalities

$0 < q < p - 1$

For a ball $B \subset \mathbf{R}^n$, denote by $\kappa(B)$ the least constant in the localized weighted norm inequality for Wolff potentials,

$$||\mathcal{W}_p \nu||_{L^q(d\sigma_B)} \le \kappa(B)\, ||\nu||^{\frac{1}{p-1}}, \quad \forall \nu \in M_+(\mathbf{R}^n). \tag{7}$$

Here $\sigma_B = \sigma|_B$ is σ restricted to a ball B; $||\nu|| = \nu(\mathbf{R}^n)$.

Equivalently, $\varkappa(B)$ can be used in place of $\kappa(B)$. Here $\varkappa(B)$ is the least constant in the localized weighted norm inequality for the p-Laplacian,

$$\left(\int_B |\varphi|^q \, d\sigma \right)^{\frac{1}{q}} \le \varkappa(B)\, ||\Delta_p \varphi||^{\frac{1}{p-1}}_{L^1(\mathbf{R}^n)}, \tag{8}$$

for all smooth test functions φ such that $-\Delta_p \varphi \ge 0$, $\liminf_{x\to\infty} \varphi(x) = 0$.

Intrinsic nonlinear potentials

$0 < q < p-1$

A new (**intrinsic**) nonlinear potential $\mathcal{K}_{p,q}\sigma$ was introduced in [Cao-Verbitsky '17]

$$\mathcal{K}_{p,q}\sigma(x) = \int_0^\infty \left(\frac{[\varkappa(B(x,r))]^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{9}$$

These potentials are closely related to solutions of $-\Delta_p u \geq \sigma u^q$.

A fractional version is defined by

$$\mathcal{K}_{p,q,\alpha}\sigma(x) = \int_0^\infty \left(\frac{[\kappa(B(x,r))]^{\frac{q(p-1)}{p-1-q}}}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r},$$

where $\kappa(B)$ is the constant in the inequality

$$||\mathcal{W}_{\alpha,p}\nu||_{L^q(d\sigma_B)} \leq \kappa(B)\, ||\nu||^{\frac{1}{p-1}}.$$

Nonlinear potentials and quasilinear equations

Denote by U a positive (p-superharmonic) solution to

$$-\Delta_p U = \mu, \qquad \liminf_{|x| \to +\infty} U(x) = 0.$$

Then [Kilpeläinen-Malý '94]

$$C_1 \mathcal{W}_p \mu(x) \leq U(x) \leq C_2 \mathcal{W}_p \mu(x), \quad \forall x \in \mathbf{R}^n. \tag{10}$$

A solution U exists if and only if $\mathcal{W}_p \mu \not\equiv \infty$, or equivalently

$$\int_1^\infty \left(\frac{\mu(B(0,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

In particular, U is **bounded** in $\mathbf{R}^n$ if and only if $\mathcal{W}_p \mu \in L^\infty(\mathbf{R}^n)$.

Problem: For which μ do we have $U \in BMO(\mathbf{R}^n)$?

This question is addressed below, requires **gradient estimates**.

Finite energy solutions and Wolff's inequality

Denote by $\dot{W}^{1,p}(\mathbf{R}^n)$ $(1 < p < n)$ the ***homogeneous*** Sobolev (Dirichlet) space, the closure of $C_0^\infty(\mathbf{R}^n)$ in the norm $||\nabla u||_{L^p(\mathbf{R}^n)}$. The dual space $\dot{W}^{-1,p'}(\mathbf{R}^n) = [\dot{W}^{1,p}(\mathbf{R}^n)]^*$.

If $-\Delta_p U = \mu$, then $U \in \dot{W}^{1,p}(\mathbf{R}^n)$ if and only if $\mu \in \dot{W}^{-1,p'}(\mathbf{R}^n)$ (that is, μ has finite p-energy $||\mu||^{p'}_{\dot{W}^{-1,p'}}$) and

$$||\nabla U||^p_{L^p(\mathbf{R}^n)} \approx \int_{\mathbf{R}^n} (\mathcal{W}_p \mu)\, d\mu \approx \int_{\mathbf{R}^n} (\mathbf{I}_1 \mu)^{p'} dx < \infty.$$

This is a special case of **Wolff's inequality** [Hedberg-Wolff '83]

$$||\mu||^{p'}_{\dot{W}^{-\alpha,p'}(\mathbf{R}^n)} \approx \int_{\mathbf{R}^n} (\mathcal{W}_{\alpha,p} \mu)\, d\mu \approx \int_{\mathbf{R}^n} (\mathbf{I}_\alpha \mu)^{p'} dx,$$

for $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, the constants depend only on p, α, n.

Finite energy solutions to $-\Delta_p u = \sigma u^q + \mu$

$q > p - 1$

We first consider finite energy solutions to (1).

Theorem (Phuc-Verbitsky '09, '20)

Let $1 < p < n$ and $q > p - 1$. Let $\mu, \sigma \in M_+(\mathbf{R}^n)$. There exists a solution $u \in \dot{W}^{1,p}(\mathbf{R}^n)$, $u \geq 0$, to (1) if and only if

$$(a) \quad \mathcal{W}_p[(\mathcal{W}_p\mu)^q d\sigma] \leq C\, \mathcal{W}_p\mu,$$

$$(b) \quad \int_{\mathbf{R}^n} (\mathcal{W}_p\mu)\, d\mu < +\infty.$$

Remarks. 1. Condition (a) holds with a small constant $c(p, q, n)$ in the *if* part, and a larger constant $C(p, q, n)$ in the *only if* part.

2. Condition (b) means that μ has finite p-energy.

Finite energy solutions to $-\Delta_p u = \sigma u^q + \mu$

$0 < q < p - 1$

Theorem (Seesanea-Verbitsky '20)

Let $1 < p < n$ *and* $0 < q < p - 1$. *There exists a positive solution* $u \in \dot{W}^{1,p}(\mathbf{R}^n)$ *to the equation* $-\Delta_p u = \sigma u^q + \mu$ *if and only if*

$$(a) \quad \int_{\mathbf{R}^n} (\mathcal{W}_p \sigma)^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < +\infty,$$

$$(b) \quad \int_{\mathbf{R}^n} (\mathcal{W}_p \mu) \, d\mu < +\infty.$$

Remarks. 1. Condition (a) is equivalent to [Cascante-Ortega-V. '00]

$$\Big(\int_{\mathbf{R}^n} |\varphi|^{1+q} \, d\sigma \Big)^{\frac{1}{1+q}} \leq C \, ||\nabla \varphi||_{L^p(\mathbf{R}^n)}, \quad \forall \varphi \in C_0^\infty(\mathbf{R}^n).$$

2. There is no interaction between μ and σ in conditions (a)&(b).

Hölder continuous, BMO solutions to $-\Delta_p U = \mu$

Consider the space of functions $u \in L^1_{\text{loc}}(\Omega)$ such that [Campanato '63]

$$\frac{1}{|B|} \int_B |u - \bar{u}_B| dx \le C\, |B|^{\frac{\alpha}{n}}, \quad \forall\, B \subset \Omega.$$

Here $u \in BMO(\Omega)$ if $\alpha = 0$; u is α-Hölder continuous if $\alpha \in (0, 1]$.

Let $1 < p < n$, $0 < \alpha < 1$, and $\mu \in M_+(\mathbf{R}^n)$. A positive α-Hölder continuous solution U to $-\Delta_p U = \mu$ exists if and only if

$$\mu(B(x,R)) \le C\, R^{n-p+\alpha(p-1)}, \quad \forall x \in \mathbf{R}^n, \ R > 0, \qquad (11)$$

locally [Kilpeläinen-Zhong '02]; globally in $\mathbf{R}^n$ if $\mathcal{W}_{1,p}\mu \not\equiv \infty$.

The case $\alpha = 0$ corresponds to $U \in BMO(\mathbf{R}^n)$, requires new methods.
Difficult range: $1 < p \le 2 - \frac{1}{n}$. For $p = 2$, we have (if $\mathbf{I}_2\mu \not\equiv \infty$):
$U = \mathbf{I}_2\mu \in BMO(\mathbf{R}^n) \Leftrightarrow \mu(B(x,R)) \le C\, R^{n-2}$ [D. Adams '75].

BMO solutions to $-\Delta_p U = \mu$

The following is based on Caccioppoli estimates, and pointwise/integral **gradient estimates** [Duzaar-Mingione '10], [Adimurthi-Phuc '14].

Theorem (Phuc-Verbitsky '20)

Let $1 < p < n$ and $\mu \in M_+(\mathbb{R}^n)$. Then there exists a nonnegative solution $U \in BMO(\mathbb{R}^n)$ to $-\Delta_p U = \mu$ if and only if

$$\mu(B(x,R)) \le C\, R^{n-p}, \quad \forall\, x \in \mathbb{R}^n, \; R > 0, \tag{12}$$

provided $\mathcal{W}_{1,p}\mu \not\equiv \infty$, i.e., $\displaystyle\int_1^\infty \left(\frac{\mu(B(0,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$

Moreover, any such solution U is in the Morrey space (for all $s < p$)

$$\int_{B(x,R)} |\nabla U|^s dy \le C\, R^{n-s}, \quad x \in \mathbb{R}^n, \; r > 0. \tag{13}$$

BMO solutions to $-\Delta_p u = \sigma u^q + \mu$

$q > p-1$

Theorem (Phuc-Verbitsky '20)

Let $1 < p < n$, $q > p-1$, *and* $\mu, \sigma \in M_+(\mathbf{R}^n)$. *Then there exists a nonnegative solution* $u \in BMO(\mathbf{R}^n)$ *to* (1) *if and only if*

$$(a) \quad \mathcal{W}_p[(\mathcal{W}_p\mu)^q d\sigma](x) \le C\, \mathcal{W}_p\mu(x),$$

$$(b) \quad \mu(B(x,R)) \le C\, R^{n-p},$$

$$(c) \quad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \le C\, R^{n-p},$$

for all $x \in \mathbf{R}^n$, $R > 0$. *Moreover,* u *satisfies the Morrey condition* (13) *for all* $s < p$.

Remark. Condition (a) holds with a small constant $c(p,q,n)$ in the *if* part, and a larger constant $C(p,q,n)$ in the *only if* part.

BMO solutions to $-\Delta_p u = \sigma u^q + \mu$

$0 < q < p - 1$

Theorem (Phuc-Verbitsky '20)

Let $1 < p < n$, $0 < q < p - 1$, $\mu, \sigma \in M_+(\mathbf{R}^n)$. *There exists a positive solution* $u \in BMO(\mathbf{R}^n)$ *to* (1) *if and only if*

$$(a) \quad \mu(B(x,R)) \le C\, R^{n-p}, \quad [\kappa(B(x,R))]^{\frac{q(p-1)}{p-1-q}} \le C\, R^{n-p},$$

$$(b) \quad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \le C\, R^{n-p},$$

$$(c) \quad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^{\frac{q(p-1)}{p-1-q}} \le C\, R^{n-p},$$

$$(d) \quad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{[\kappa(B(x,r))]^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \le C\, R^{n-p}.$$

BMO solutions to $-\Delta_p u = \sigma u^q + \mu$

$0 < q < p-1$ (continuation)

Corollary

Let $1 < p < n$ *and* $0 < q < p-1$*. Suppose* $\mu, \sigma \in M_+(\mathbf{R}^n)$*, where* $\sigma(K) \le C\, cap_p(K)$*,* $\forall\, K \subset \mathbf{R}^n$*. Then there exists a positive solution* $u \in BMO(\mathbf{R}^n)$ *to* (1) *if and only if, for all* $x \in \mathbf{R}^n$ *and* $R > 0$*,*

$$(a) \quad \mu(B(x,R)) \le C\, R^{n-p},$$

$$(b) \quad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \le C\, R^{n-p},$$

$$(c) \quad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{\rho} \right]^{\frac{q(p-1)}{p-1-q}} \le C\, R^{n-p}.$$

Remark. In the previous theorem and corollary, the solution u satisfies the Morrey condition (13), for all $s < p$.

BMO solutions to $-\Delta_p u = \sigma u^{p-1} + \mu$

$q = p - 1$

Theorem (Phuc-Verbitsky '20)

Let $1 < p < n$ *and* $q = p - 1$. *Suppose* $\mu, \sigma \in M_+(\mathbf{R}^n)$, *and*

$$\mathcal{W}_p\sigma \le C \text{ if } p > 2 \quad \text{and} \quad \mathbf{I}_2\sigma \le C \text{ if } p \le 2.$$

Then there exists a positive solution $u \in BMO(\mathbf{R}^n)$ *to* (1) *if and only if, for all* $x \in \mathbf{R}^n$ *and* $R > 0$,

$$(a) \quad \mu(B(x,R)) \le C\, R^{n-p},$$

$$(b) \quad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^{p-1} \le C\, R^{n-p}.$$

Remarks. 1. The *if* part requires the smallness of the constant in

$$\sigma(K) \le c(p,n)\, \text{cap}_p(K), \quad \forall K \subset \mathbf{R}^n.$$

2. Without the boundedness of the potentials assumption, the results are more complicated.

Capacity classes of solutions

$0 < q < p - 1$

We next characterize solutions u in the more restricted than BMO class

$$\int_K |\nabla u|^p dx \le C \operatorname{cap}_p(K), \quad \forall K \subset \mathbf{R}^n. \tag{14}$$

Theorem (Verbitsky '20)

Let $1 < p < n$, $0 < q < p - 1$, and $\sigma, \mu \in M_+(\mathbf{R}^n)$. Then there exists a positive solution u to (1) which satisfies condition (14) if and only if, for all compact sets K in $\mathbf{R}^n$,

$$\textbf{(a)} \quad \mu(K) \le C \operatorname{cap}_p(K),$$

$$\textbf{(b)} \quad \int_K (\mathcal{W}_p\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma \le C \operatorname{cap}_p(K),$$

$$\textbf{(c)} \quad \int_K (\mathcal{W}_p\mu)^q d\sigma \le C \operatorname{cap}_p(K).$$

Pointwise estimates

$q > p - 1$

Theorem (Phuc-Verbitsky '09)

Let $1 < p < n$ *and* $q > p - 1$*. There exists a solution* $u > 0$*,* $\liminf_{x \to \infty} u(x) = 0$*, to* (1) *iff* $\mathcal{W}_p \mu \not\equiv \infty$ *and*

$$\mathcal{W}_p[(\mathcal{W}_p \mu)^q d\sigma](x) \leq C \, \mathcal{W}_p \mu(x), \quad \forall \, x \in \mathbf{R}^n.$$

Moreover, there exist constants $C_1, C_2 > 0$ *such that*

$$C_1 \mathcal{W}_p \mu(x) \leq u(x) \leq C_2 \mathcal{W}_p \mu(x), \quad \forall \, x \in \mathbf{R}^n.$$

Remarks. 1. Condition (a) holds with a small constant $c(p, q, n)$ in the *if* part, and a larger constant $C(p, q, n)$ in the *only if* part.
2. The lower estimate holds for all solutions u, whereas the upper one holds for the minimal solution.

Pointwise estimates

$q = p - 1$

Theorem (Jaye-Verbitsky '12)

Let $1 < p < n$ and $0 < q < p - 1$. Suppose $\mu, \sigma \in M_+(\mathbf{R}^n)$, and

$$\mathcal{W}_p\sigma \le C \text{ if } p > 2 \quad \text{and} \quad \mathbf{I}_2\sigma \le C \text{ if } p \le 2.$$

Then there exists a positive solution u to (1) if and only if $\mathcal{W}_p\mu \not\equiv \infty$, and

$$C_1 \mathcal{W}_p\mu(x) \le u(x) \le C_2 \mathcal{W}_p\mu(x), \quad \forall\, x \in \mathbf{R}^n,$$

for positive constants $C_1, C_2 > 0$.

Remarks. 1. The *if* part requires the smallness of the constant in

$$\sigma(K) \le c(p,n)\, \text{cap}_p(K), \quad \forall K \subset \mathbf{R}^n.$$

2. The lower estimate holds for all solutions, and the upper one for the minimal solution.

Pointwise estimates of Brezis-Kamin type

$0 < q < p - 1$

We first discuss global estimates of Brezis-Kamin type.

Theorem (Cao-Verbitsky '16, Verbitsky '20)

Let $1 < p < n$, $0 < q < p - 1$. *Suppose* $\sigma(K) \le C\, cap_p(K)$, $\forall\, K \subset \mathbf{R}^n$. *Then there exists a positive solution* u *to* (1) *such that* $\liminf_{|x| \to +\infty} u(x) = 0$ *if and only if* $\mathcal{W}_p\mu + \mathcal{W}_p\sigma \not\equiv \infty$, *and*

$$C_1 \left(\mathcal{W}_p\mu + (\mathcal{W}_p\sigma)^{\frac{p-1}{p-1-q}} \right) \le u \le C_2 \left(\mathcal{W}_p\mu + \mathcal{W}_p\sigma + (\mathcal{W}_p\sigma)^{\frac{p-1}{p-1-q}} \right)$$

Remarks. 1. [Brezis-Kamin '92] for ***bounded*** solutions, $\mu = 0$, $p = 2$.
2. A solution $u \in L^\infty(\mathbf{R}^n)$ if and only if $\mathcal{W}_p\mu + \mathcal{W}_p\sigma \in L^\infty(\mathbf{R}^n)$.
3. Lower/upper bounds do not match, upper bound for minimal solutions.

Existence of general solutions to $-\Delta_p u = \sigma u^q + \mu$

$0 < q < p-1$

Theorem (Cao-Verbitsky '17, Verbitsky '20)

Let $1 < p < n$ *and* $0 < q < p-1$. *Let* $\mu, \sigma \in M_+(\mathbf{R}^n)$. *Then there exists a positive solution* u *to (1) such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if the following conditions hold:*

$$\int_1^\infty \left(\frac{\mu(B(0,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} + \int_1^\infty \left(\frac{\sigma(B(0,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty, \qquad (15)$$

$$\int_1^\infty \left(\frac{[\varkappa(B(0,r))]^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty. \qquad (16)$$

Remark. If $\sigma(K) \le C\,\mathrm{cap}_p(K)$, then $[\varkappa(B)]^{\frac{q(p-1)}{p-1-q}} \le C_1\,\sigma(B)$, and condition (16) is redundant. Moreover, $\mathcal{K}_{p,q}\sigma(x) \le C_2 \mathcal{W}_p\sigma(x)$.

Bilateral pointwise estimates in the general case

$0 < q < p - 1$

Theorem (Cao-Verbitsky '17, Verbitsky '20)

Moreover, the following bilateral global pointwise estimates hold for the (minimal) solution $u > 0$ *to* $-\Delta_p u = \sigma u^q + \mu$ *on* $\mathbf{R}^n$*:*

$$u(x) \approx \mathcal{W}_p \mu(x) + (\mathcal{W}_p \sigma(x))^{\frac{p-1}{p-1-q}} + \mathcal{K}_{p,q}\sigma(x). \tag{17}$$

Recall that $\mathcal{K}_{p,q}\sigma$ is defined by

$$\mathcal{K}_{p,q}\sigma(x) = \int_0^\infty \left(\frac{[\varkappa(B(x,r))]^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{18}$$

Remarks. 1. The lower bound holds for all positive solutions, whereas the upper bound holds for the (unique) minimal solution.
2.The constants of equivalence depend only on p, q, n.

Local $W^{1,p}$ solutions to $-\Delta_p u = \sigma u^q$

$0 < q < p-1$

If we wish to find a nontrivial solution $u \in W^{1,p}_{\text{loc}}(\mathbf{R}^n)$ to $-\Delta_p u = \sigma u^q$ (for simplicity let $\mu = 0$), then an additional local version of the condition for finite energy solutions is needed:

$$\int_B (\mathcal{W}_{1,p}\sigma_B)^{\frac{(1+q)(p-1)}{p-1-q}}\, d\sigma < \infty, \tag{19}$$

for all balls B in $\mathbf{R}^n$.

Theorem (Cao-Verbitsky '17)

Under the assumptions of the previous theorem, there exists a nontrivial solution $u \in W^{1,p}_{loc}(\mathbf{R}^n)$ to $-\Delta_p u = \sigma u^q$ such that $\liminf_{|x|\to+\infty} u(x) = 0$ if and only if conditions (15), (16) and (19) hold. Moreover, pointwise estimates (17) hold for the minimal solution.

Remark. Condition (16) can be dropped if $\sigma(K) \le C\, \mathrm{cap}_p(K)$.

Wolff's inequality for intrinsic potentials

$0 < q < p - 1$

The following form of Wolff's inequality holds for intrinsic potentials.

Theorem (Verbitsky '20)

Let $1 < p < n$, $0 < q < p - 1$, $\frac{n(p-1)}{n-p} < r < \infty$. *Then*

$$\|\mathcal{K}_{p,q}\sigma\|^r_{L^r(\mathbf{R}^n)} \approx \int_{\mathbf{R}^n} \sup_{\rho>0} \left(\frac{[\kappa(B(x,\rho))]^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{r}{p-1}} dx, \qquad (20)$$

where the constants of equivalence depend only on p, q, r*, and* n.

Remark. If $p \geq n$, or $1 < p < n$, $0 < r \leq \frac{n(p-1)}{n-p}$, and $\mathcal{K}_{p,q}\sigma \in L^r(\mathbf{R}^n)$, then $\sigma = 0$.

L^r solutions to $-\Delta_p u = \sigma u^q + \mu$

$0 < q < p - 1$

Corollary (Verbitsky '20)

Let $1 < p < n$ *and* $0 < q < p - 1$. *Suppose that* $\frac{n(p-1)}{n-p} \le r < \infty$. *Then there exists a positive solution* $u \in L^r(\mathbf{R}^n)$ *to* (1) *if and only if conditions* (15), (16) *hold, and for all* $R > 0$

$$\int_{\mathbf{R}^n} \sup_{\rho>0} \left(\frac{\mu(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{r}{p-1}} dx < \infty,$$

$$\int_{\mathbf{R}^n} \sup_{\rho>0} \left(\frac{[\kappa(B(x,\rho))]^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{r}{p-1}} dx < \infty.$$

If $p \ge n$, or $1 < p < n$ and $0 < r \le \frac{n(p-1)}{n-p}$, then $\sigma = 0$, $\mu = 0$.

Local L^r solutions to $-\Delta_p u = \sigma u^q + \mu$

$0 < q < p-1$

Here is a local version of the previous result.

Corollary (Verbitsky '20)

Let $1 < p < n$ *and* $0 < q < p-1$. *Suppose that* $\frac{n(p-1)}{n-p} \le r < \infty$. *Then there exists a nontrivial solution* $u \in L^r_{\mathrm{loc}}(\mathbf{R}^n)$ *to* (1) *if and only if conditions* (15), (16) *hold, and for all* $R > 0$,

$$\int_{B(0,R)} \sup_{0<\rho<R} \left(\frac{\mu(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{r}{p-1}} dx < \infty,$$

$$\int_{B(0,R)} \sup_{0<\rho<R} \left(\frac{[\kappa(B(x,\rho))]^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{r}{p-1}} dx < \infty,$$

If $0 < r < \frac{n(p-1)}{n-p}$, *then there exists a nontrivial solution* $u \in L^r_{\mathrm{loc}}(\mathbf{R}^n)$ *to* (1) *whenever conditions* (15), (16) *hold.*

Local L^r solutions to $-\Delta_p u = \sigma u^q + \mu$

$0 < q < p-1$ (continuation)

The following corollary is deduced under the additional assumption

$$\sigma(K) \le C\, \mathrm{cap}_p(K), \quad \forall\, K \subset \mathbb{R}^n. \tag{21}$$

Corollary

Let $1 < p < n$, $0 < q < p-1$, *and* $\frac{n(p-1)}{n-p} \le r < \infty$. *If* $\sigma \in M_+(\mathrm{R}^n)$ *satisfies condition* (21), *then there exists a positive solution* $u \in L^r_{\mathrm{loc}}(\mathrm{R}^n)$ *to* (1) *if and only if conditions* (15), (16) *hold, and for all* $R > 0$,

$$\int_{B(0,R)} \sup_{0<\rho<R} \left(\frac{\mu(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{r}{p-1}} dx < \infty.$$

Remark. If $0 < r < \frac{n(p-1)}{n-p}$, then every p-superharmonic function lies in $L^r_{\mathrm{loc}}(\mathrm{R}^n)$. Hence, there exists a positive solution $u \in L^r_{\mathrm{loc}}(\mathrm{R}^n)$ to (1) whenever conditions (15), (16) hold.

More general $\mathcal{A}$-Laplace operators

Let us assume that $\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the following structural assumptions:

$$x \to \mathcal{A}(x, \xi) \quad \text{is measurable for all } \xi \in \mathbf{R}^n,$$

$$\xi \to \mathcal{A}(x, \xi) \quad \text{is continuous for a.e. } x \in \mathbf{R}^n,$$

and there are constants $0 < \alpha \le \beta < \infty$, so that for a.e. x in $\mathbf{R}^n$, all ξ in $\mathbf{R}^n$

$$\langle \mathcal{A}(x, \xi), \xi \rangle \ge \alpha |\xi|^p, \quad |\mathcal{A}(x, \xi)| \le \beta |\xi|^{p-1}, \tag{22}$$

$$\langle \mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2), \xi_1 - \xi_2 \rangle > 0 \quad \text{if } \xi_1 \ne \xi_2 \tag{23}$$

These conditions suffice for the [Kilpeläinen-Malý '94] pointwise estimates.

Renormalized solutions

Consider the equation

$$- \operatorname{div}\mathcal{A}(x, \nabla u) = \mu \quad \text{in} \quad \Omega, \tag{24}$$

where $\mu \in M_+(\Omega)$, and $\Omega \subseteq \mathbf{R}^n$ is an open set. Let us use the decomposition $\mu = \mu_0 + \mu_s$: μ_0 is absolutely continuous, and μ_s is singular with respect to p-capacity. Let $T_k(s) = \max\{-k, \min\{k, s\}\}$. Then u is a ***local renormalized solution*** to (24) if, for all $k > 0$, $T_k(u) \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $u \in L^{(p-1)s}_{\mathrm{loc}}$ for $1 \leq s < \frac{n}{n-p}$, $Du \in L^{(p-1)r}_{\mathrm{loc}}(\Omega)$ for $1 \leq r < \frac{n}{n-1}$, and

$$\int_\Omega \langle \mathcal{A}(x, Du), Du \rangle \, h'(u) \, \phi \, dx + \int_\Omega \langle \mathcal{A}(x, Du), \nabla \phi \rangle h(u) \, \phi \, dx$$

$$= \int_\Omega h(u) \, \phi \, d\mu_0 + h(+\infty) \int_\Omega \phi \, d\mu_s,$$

for all $\phi \in C_0^\infty(\Omega)$, and $h \in W^{1,\infty}(\mathbf{R})$, h' is compactly supported.

$\mathcal{A}$-Laplace operators

In the above definition Du is given by: $Du = \lim_{k\to\infty} \nabla(T_k(u))$.

Every $\mathcal{A}$-superhamonic function is locally a renormalized solution, and conversely, every local renormalized solution has an $\mathcal{A}$-superharmonic representative. One can work either with local renormalized solutions, or equivalently with potential theoretic solutions.

For finite energy solutions $u \in \dot{W}_0^{1,p}(\Omega)$, $Du = \nabla u$, and μ is absolutely continuous with respect to the p-capacity.

Basic facts of potential theory remain true for the $\mathcal{A}$-Laplacian.

Remark. More restrictions on $\mathcal{A}(x, \xi)$ are needed for BMO solutions and gradient estimates. (Not necessary for most existence theorems, pointwise estimates of solutions, and finite energy estimates.)

$\mathcal{A}$-Laplace operators (continuation)

Our results on positive BMO solutions to the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = \sigma u^q + \mu \quad \text{in } \mathbf{R}^n, \tag{25}$$

with $\sigma, \mu \in M_+(\mathbf{R}^n)$, remain valid under the following assumptions on $\mathcal{A}(x,\xi)$:

$\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is measurable in x for every ξ, continuous in ξ for a.e. x, $\mathcal{A}(x,0) = 0$ for a.e. $x \in \mathbf{R}^n$, and there exist positive constants α_0, β_0 such that

$$\langle \mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2), \xi_1 - \xi_2 \rangle \geq \alpha_0 (|\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \tag{26}$$

$$|\mathcal{A}(x,\xi)| \leq \beta_0 |\xi|^{p-1}. \tag{27}$$

BMO solutions to $-\mathrm{div}\,\mathcal{A}(x, \nabla u) = \mu$

Our previous results on BMO solutions for (1) involving the p-Laplacian extend to equations with the $\mathcal{A}$-Laplacian under assumptions (26), (27).

In the case $2 - \frac{1}{n} < p < n$, one can use pointwise gradients estimates of [Mingione '11], [Duzaar-Mingione '10], [Kuusi-Mingione '13].

In the case $\frac{3n-2}{2n-1} < p \le 2 - \frac{1}{n}$, we use recent integral gradient estimates developed in [Adimurthi-Phuc '15], [Adimurthi-Mengesha-Phuc '18], [Phuc '14] , and in the most delicate case $1 < p \le \frac{3n-2}{2n-1}$ obtained very recently in [Nguyen-Phuc '20].

It is enough to prove the following theorem for BMO solutions in the special case $\sigma = 0$. Let $\mu \in M_+(\mathbf{R}^n)$. Consider nonnegative $\mathcal{A}$-superharmonic solutions (equivalently local renormalized solutions) to the equation

$$-\,\mathrm{div}\,\mathcal{A}(x, \nabla U) = \mu \text{ in } \mathbf{R}^n. \tag{28}$$

BMO solutions to $-\mathrm{div}\, \mathcal{A}(x, \nabla U) = \mu$

A local version of the following theorem was originally established in [Mingione '07] for $p > 2$. The case $p = 2$ is due to [D. Adams '75].

Theorem (Phuc-Verbitsky '20)

Let $1 < p < n$ *and* $\mu \in M_+(\mathbf{R}^n)$. *Let* U *be a positive solution to* (28), *under assumptions* (26), (27) *on* $\mathcal{A}$. *Suppose that*

$$\mu(B(x,R)) \le C\, R^{n-p}, \quad \forall\, x \in \mathbf{R}^n,\ R > 0. \tag{29}$$

Then $U \in BMO(\mathbf{R}^n)$, *and* $\|U\|_{BMO(\mathbf{R}^n)} \le c\, C^{\frac{1}{p-1}}$, *where* $c = c(p, n, \alpha_0, \beta_0)$ *is a positive constant.*

The converse statement $U \in BMO(\mathbf{R}^n) \Rightarrow$(29) holds [Verbitsky '20] under (22), (23). This yields a criterion of existence for $U \in BMO(\mathbf{R}^n)$.

Corollary. For $1 < p < n$, $\mu \in M_+(\mathbf{R}^n)$, and $\mathcal{A}$ that obeys (26), (27), there exists a solution $U \ge 0$ to (28), $U \in BMO(\mathbf{R}^n) \Leftrightarrow$ (6)&(29) hold.